

Homework 8

Due on **Nov 4 before 1pm** on gradescope.

To receive full credit, show necessary reasoning unless it's straightforward computation.

1. (20 pts) True or false (you can simply answer T or F, no need to give justification; but try to think about why).

(a) A square matrix of size $n \times n$ is diagonalizable if and only if it has n linearly independent eigenvectors.

(b) The matrix $A = \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 2 & 1 & -2 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$ has an eigenvalue $\lambda = 4$, and this eigenvalue $\lambda = 4$ has only one eigenvector.

(c) All eigenvectors of a square matrix form a vector space.

(d) Any $A \in \mathbb{R}^{n \times n}$ always has n complex eigenvalues including repeated ones.

(e) If $\text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is an eigenspace for a matrix $A \in \mathbb{R}^{3 \times 3}$ with eigenvalue $\lambda = 3$. Then

$$A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}.$$

(f) For $A \in \mathbb{R}^{n \times n}$, if $AV = VD$ for some invertible $V \in \mathbb{R}^{n \times n}$ and a diagonal $D \in \mathbb{R}^{n \times n}$, then A has n eigenvalues including repeated ones.

(g) For $A \in \mathbb{R}^{n \times n}$, if $AV = VD$ for some invertible $V \in \mathbb{R}^{n \times n}$ and a diagonal $D \in \mathbb{R}^{n \times n}$, then the geometrical multiplicity is equal to the

algebraic multiplicity for any eigenvalue of A .

(h) Any noninvertible $A \in \mathbb{R}^{n \times n}$ has at least one real eigenvalue.

(i) If $A \in \mathbb{R}^{n \times n}$ is singular (not invertible), then $n - \text{rank}(A)$ is equal to the dimension of one of its eigenspaces.

(j) The dimensions of all eigenspaces of a matrix $A \in \mathbb{R}^{n \times n}$ must sum to n .

2. (20 pts) Determine whether the matrix is diagonalizable or not. If yes, find the diagonalizable in the form $A = VDV^{-1}$. Use complex matrices if needed.

$$A = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}.$$

3. (20 pts) Determine whether the matrix is diagonalizable or not. If yes, find the diagonalizable in the form $A = VDV^{-1}$. Use complex matrices if needed.

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}.$$

4. (20 pts) Find e^A for $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Multiply all matrices to find e^A as a single matrix. Use Euler's formula $e^{i\theta} = \cos \theta + i \sin \theta$ to simplify your answer if you feel like to.

5. (20 pts) We can find solutions to an ordinary differential equation

$$y'''(t) + y''(t) - y'(t) - y(t) = 0$$

following these steps:

(a) Write it as a first order system $\frac{d}{dt}\vec{u}(t) = A\vec{u}(t)$ with

$$\vec{u}(t) = \begin{pmatrix} y(t) \\ y'(t) \\ y''(t) \end{pmatrix}.$$

Find what A is.

(b) Find basis vectors for each eigenspace of A .

- (c) Let \vec{v} be an eigenvector with eigenvalue λ , then $\vec{u}(t) = e^{\lambda t}\vec{v}$ is a solution. Use all the basis vectors you can find to write the solutions for \vec{u} , and plug it into both sides of $\frac{d}{dt}\vec{u}(t) = A\vec{u}(t)$ to verify that they are solutions.
- (d) Plug $y(t)$ that you get from $\vec{u}(t)$ back into the original third order differential equation to verify that it is a solution.