MA 351 Section 041/042
Fall 2021

Homework 9

Due on **Dec 2 before 1pm** on gradescope.

To receive full credit, show necessary reasoning unless it's straight-forward computation.

1. (20 pts) True or false (you can simply answer T or F, no need to give justification; but try to think about why).

   (a) If a real square matrix of size $n \times n$ has n real orthonormal eigenvectors, then these eigenvectors are also its singular vectors.

   (b) If a real square matrix $A = VDV^T$, where V is a real matrix with orthonormal columns and the diagonal matrix D has diagonal entries $d_i \in \mathbb{R}$, then the singular values of A are $\sigma_i(A) = |d_i|$.

   (c) For a real symmetric $A \in \mathbb{R}^{n\times n}$, $\vec{x}^T A\vec{x} < 0$ for any nonzero vector $\vec{x} \in \mathbb{R}$ if and only if all eigenvalues of A are negative.

   (d) For a real symmetric matrix A, if its singular values are also its eigenvalues, then it is positive semi-definite.

   (e) A real square matrix A is invertible if and only if all its singular values are positive.

   (f) If $A \in \mathbb{R}^{n\times n}$ is normal, then its eigenvalues must be equal to its singular values.

   (g) The rank of $A \in \mathbb{R}^{n\times n}$ is equal to the number of its nonzero singular values.

   (h) If $A \in \mathbb{R}^{n\times n}$ is real symmetric with eigenvalues λ_i $(i = 1, \cdots, n)$, then the Frobenius norm of A is equal to $\sqrt{\sum_{i=1}^{n} |\lambda_i|^2}$.

   (i) For $A \in \mathbb{R}^{n\times n}$ with entries a_{ij} $(i, j = 1, \cdots, n)$ and singular values

σ_i $(i = 1, \cdots, n)$, the following holds:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2 = \sum_{i=1}^{n} \sigma_i^2.$$

(j) If $A \in \mathbb{R}^{n\times n}$ has orthonormal columns, then it is diagonalizable.

2. (30 pts) Consider the matrix

$$A = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

(a) (10 pts) Find all eigenvalues, their algebraic multiplicity and geometrical multiplicity, and basis vectors for all eigenspaces.

(b) (10 pts) For this particular matrix, there is one eigenvalue λ_2 for which geometrical multiplicity is less than algebraic multiplicity. This ensures existence of one generalized eigenvector: let v be its eigenvector, then find the generalized eigenvector u defined as solution to the nonhomogeneous linear system $(A - \lambda_2 I)u = v$.

(c) (10 pts) For this particular matrix, there are two distinct eigenvalues λ_1 and λ_2. Let v_1 be eigenvector for λ_1. Form a matrix $V = [v_1\ v\ u]$. Then by the definition of eigenvectors and generalized eigenvectors, we have

$$AV = [Av_1\ Av\ Au] = [\lambda_1 v_1\ \lambda_2 v\ \lambda_2 u + v] = [v_1\ v\ u] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix}.$$

Here $J = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix}$ is called Jordan Form of A. Find the explicit expression of J, V, V^{-1} and verify that $A = VJV^{-1}$ (and this is what eigenvalue decomposition looks like for a nondiagonalizable matrix).

3. (30 pts) Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find its SVD $A = U\Sigma V^T$ by computing σ_i^2 as eigenvalues of AA^T (or A^TA), computing columns u_i of U as orthonormal eigenvectors of AA^T and columns v_i of V as orthonormal eigenvectors of A^TA. **And order them so that** $Av_i = \sigma_i u_i$**.** Finally verify that $A = U\Sigma V^T$.

4. (10 pts) Recall that (defined in HW#1) the sum of all diagonal entries of A is called the trace of A, often denoted as tr(A) or trace(A). Assume $A \in \mathbb{R}^{3\times 3}$ have entries a_{ij}. Verify that

$$tr(A^TA) = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}^2.$$

5. (10 pts) Let $A \in \mathbb{R}^{3\times 3}$ be a real symmetric matrix, then it has 3 real eigenvalues including repeated ones: $\lambda_1, \lambda_2, \lambda_3$. Show that

$$tr(A) = \sum_{i=1}^{3} \lambda_i, \quad tr(A^TA) = \sum_{i=1}^{3} |\lambda_i|^2.$$

Hint: use diagonalization. The trace satisfies $tr(ABC) = tr(BCA)$ for any three matrices A, B, C of suitable sizes.