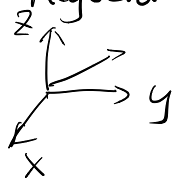
Elementary Linear Algebra

vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ 

$\mathbb{R}^3$ Matrix $\begin{bmatrix} 1.1 & 2.0 & 3 \\ 4 & 5 & 6i \end{bmatrix} \in \mathbb{C}^{2\times 3}$

2×3 $\quad i = \sqrt{-1}$

Notations (Informal Def)

- A field F is a set of numbers for which $+ \ - \ \times \ \div$ are defined.

  Ex: 1) All real numbers $\mathbb{R}$
  2) All complex numbers $\mathbb{C}$
  3) All rational numbers $\mathbb{Q}$

  All natural numbers $\mathbb{N}$ ✗

- Given a field F

  1) Col vector in F $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ $a_i \in F$

     F^n $\quad F^{n\times 1}$

  2) row vector in F $(a_1, a_2, \cdots, a_n)$

     $F^{1\times n}$

  3) matrices in F $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m\times n} = A \in F^{m\times n}$

     a_{ij}: j — col index, i — row index

     $F^{m\times n}$

ith row $\leftarrow \begin{pmatrix} a_{i1} & \cdots & a_{in} \end{pmatrix}_{m\times n}$ $\qquad \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}_{m\times n}$ $\to$ j-th col

A_{ij} denotes (i,j) entry.

$A, B \in F^{m \times n}$ $\begin{cases} \text{addition}: & (A+B)_{ij} = A_{ij} + B_{ij} \\ \text{scalar multiplication}: & (cA)_{ij} = cA_{ij} \end{cases}$

$c \in F$

4) Polynomials with coefficients in F.

$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad a_i \in F.$

$q(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0, \quad b_i \in F$

$p(x) + q(x) = (a_n + b_n) x^n + \cdots + (a_0 + b_0)$

(Abstract) Vectors	Vector Space	Addition	Scalar Multiplication
$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ $a_i \in F$	All col vectors of size n F^n		$c \in F$ $c\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} ca_1 \\ \vdots \\ ca_n \end{bmatrix}$
$[a_1 \cdots a_n]$	All row vectors of size n		
$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$	All $m \times n$ matrices		$(cA)_{ij} = cA_{ij}$
$p(x) = a_n x^n + \cdots + a_0$	All polynomials	$p(x) + q(x)$	$c \in F$ $cq(x) = ca_n x^n + \cdots + ca_0$

<u>Definition</u> A vector space (or linear space) V over a field F is a set with two operations (addition and scalar mul) such that

I. V is closed under two operations

1) $\forall x, y \in V$, $\exists$ a unique $x \oplus y \in V$.

for any — there is

2) $\forall x \in V$, $\forall a \in F$, $\exists$ a unique $a \cdot x \in V$.

II. The following holds:

(VS1) $\forall x, y \in V$, $x \oplus y = y \oplus x$.

(VS2) $\forall x, y, z \in V$, $(x \oplus y) \oplus z = x \oplus (y \oplus z)$.

zero vector (VS3) $\exists$ an element in V denoted by $\vec{0}$ s.t. $x \oplus \vec{0} = x$, $\forall x \in V$.

negative element (VS4) $\forall x \in V$, $\exists y \in V$ s.t. $x \oplus y = \vec{0}$.

(VS5) $\forall x \in V$, $1 \in F$, $1 \cdot x = x$.

(VS6) $\forall a, b \in F$, $(ab)x = a(bx)$, $\forall x \in V$.

(VS7) $\forall a \in F$, $\forall x, y \in V$, $a(x \oplus y) = ax \oplus ay$.

(VS8) $\forall a, b \in F$, $\forall x \in V$, $(a+b)x = ax \oplus bx$.

Example 1: $V = \{$all real-valued single-variable continuous functions: $f(x)$, $x \in \mathbb{R}\}$ is a vector space over the field $\mathbb{R}$.

Verify definition:

I. $\forall f(x), g(x) \in V$, $\forall a \in \mathbb{R}$

$f(x) + g(x) \in V$, $af(x) \in V$.

II. (VS1) and (VS2) are trivial

(VS3): $\vec{0}$ is $f(x) \equiv 0$.

(VS4): $f(x) + (-f(x)) = \vec{0}$.

(VS5): $1 \cdot f(x) = f(x)$

(VS6), (VS7), (VS8) are trivial.

Example 2: (Example 6 P11)

$S = \{(a_1, a_2),\ a_i \in \mathbb{R}\}$

$\forall (a_1, a_2), (b_1, b_2) \in S,\ \forall c \in \mathbb{R}.$

Define $(a_1, a_2) \oplus (b_1, b_2) = (a_1 + b_1, a_2 - b_2)$

Define $c(a_1, a_2) = (ca_1, ca_2)$

I. closed under two operations.

II. (VS1) does not hold:

$(b_1, b_2) \oplus (a_1, a_2) = (b_1 + a_1, b_2 - a_2)$

Example 3: $S = \{0, 1, 2\}$ S is not a vector space because $\mathbb{N}$ is not a field.

$\forall a \in \mathbb{N}$ is not a field

$\forall x, y \in S$, define $x \oplus y = \text{mod}(x+y, 3) \in S$

$a \cdot x = \text{mod}(ax, 3) \in S.$

I. closed.

II. All 8 conditions hold

(VS3) $0 \oplus 0 = 0$

$1 \oplus 0 = \text{mod}(1+0, 3) = 1$

$2 \oplus 0 = \text{mod}(2+0, 3) = 2$

$\Rightarrow \vec{0} = 0$

(VS4) $x \oplus y = \vec{0}$

$0 \oplus 0 = 0$

$1 \oplus 2 = \text{mod}(1+2, 3) = 0$

$2 \oplus 1 = \text{mod}(2+1, 3) = 0$

(VS 8) $(a+b)x = ax \oplus bx$

$\forall a, b \in \mathbb{N}$,

$$(a+b)\cdot 2 = a\cdot 2 \oplus b\cdot 2$$

$$LHS = (a+b)\cdot 2 = \mathrm{mod}((a+b)2, 3)$$
$$= \mathrm{mod}(2a+2b, 3)$$

$$RHS = (a\cdot 2) \oplus (b\cdot 2)$$
$$= \mathrm{mod}(2a, 3) \oplus \mathrm{mod}(2b, 3)$$
$$= \mathrm{mod}(\underline{\mathrm{mod}(2a, 3) + \mathrm{mod}(2b, 3)}, 3)$$
$$= \mathrm{mod}(\mathrm{mod}(2a+2b, 3), 3)$$
$$= \mathrm{mod}(2a+2b, 3)$$

Theorem 1.1 $\forall x, y, z \in V$ where V is a vector space

$$x \oplus z = y \oplus z \Rightarrow x = y.$$

Proof: (VS4) $\Rightarrow \exists v \in V$ s.t. $z \oplus v = \vec{0}$

$$x = x \oplus \vec{0} = x \oplus (z \oplus v) = (x \oplus z) \oplus v$$
$$\|$$
$$y = y \oplus \vec{0} = y \oplus (z \oplus v) = (y \oplus z) \oplus v.$$

Corollary 1.2 $\vec{0}$ is unique in (VS3).

Proof: Assume $\exists v \in V$ s.t. $x \oplus v = x$, $\forall x \in V$.

$\Rightarrow \vec{0} \oplus v = \vec{0}$ ①

$\vec{0} \oplus x = x$, $\forall x \in V \Rightarrow \vec{0} \oplus v = v$ ②

①, ② $\Rightarrow v = \vec{0}$. #

Corollary 1.3 y in (VS4) is unique.

Proof: For any fixed $x \in V$, assume $\exists\ y_1, y_2$ s.t.

$$\begin{cases} x \oplus y_1 = \vec{0} \\ x \oplus y_2 = \vec{0} \end{cases}.$$

$$\begin{aligned} y_1 &\overset{(VS3)}{=} y_1 \oplus \vec{0} = y_1 \oplus (x \oplus y_2) \\ &\overset{(VS2)}{=} (y_1 \oplus x) \oplus y_2 \\ &\overset{(VS1)}{=} (x \oplus y_1) \oplus y_2 \\ &= \vec{0} \oplus y_2 \\ &\overset{(VS1)}{=} y_2 \oplus \vec{0} \\ &\overset{(VS3)}{=} y_2. \end{aligned}$$