① Change of Coordinate Matrix

V has two ordered bases β and γ

$Q = [I]_\beta^\gamma$ changes $[v]_\beta$ to $[v]_\gamma$:

$$[v]_\gamma = [I(v)]_\gamma = [I]_\beta^\gamma [v]_\beta = Q[v_\beta]$$

Theorem: If $Q = [I]_\beta^\gamma$, then Q is invertible and

$$Q^{-1} = [I]_\gamma^\beta$$

Proof: $\begin{cases} [v]_\beta = [I(v)]_\beta = [I]_\gamma^\beta [v]_\gamma \\ [v]_\gamma = [I]_\beta^\gamma [v]_\beta \end{cases}$ $\quad I: \overset{\gamma}{V} \longrightarrow \overset{\beta}{V}$

$$\Rightarrow I[v]_\beta = \underline{[I]_\gamma^\beta [I]_\beta^\gamma} [v]_\beta, \quad \forall v \in V.$$

(Lemma) $\Rightarrow$ $[I]_\gamma^\beta [I]_\beta^\gamma$ is identity matrix.

Lemma: $A, B \in F^{n\times n}$, if $\underline{Ax} = \underline{Bx}$, for any $\underline{x \in F^n}$ then $\underline{A = B}$.

Proof: set $x = e_i$, then $Ae_i = \begin{bmatrix} | & | & \cdots & | \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} =$ ith col of A

$\Rightarrow$ i-th col of A = ith col of B.

Remark: 1) Given β, γ, $Q = [I]_\beta^\gamma$ is invertible

2) Given β, and invertible Q, then $\exists\, \gamma$ s.t. $Q = [I]_\beta^\gamma$

Proof: $\beta = \{v_1, \dots, v_n\}$ $\quad Q = [q_{ij}]_{n\times n}$

$\gamma = \{u_1, \dots, u_n\}$ $\quad Q^{-1} = [q^{ij}]_{n\times n}$

$$[u_j]_\beta = [I]_\gamma^\beta [u_j]_\gamma = Q^{-1}\underline{[u_j]_\gamma} = Q^{-1} e_j = j\text{th col of } Q^{-1}$$

$$\Rightarrow u_j = q^{1j} v_1 + q^{2j} v_2 + \cdots + q^{nj} v_n. \qquad = \begin{bmatrix} q^{1j} \\ q^{2j} \\ \vdots \\ q^{nj} \end{bmatrix}$$

Theorem 2.23 (in Section 2.5)

$$T: \overset{\beta}{V} \longrightarrow \overset{\beta}{V} \qquad [T]_\beta$$

$$T: \overset{\gamma}{V} \longrightarrow \overset{\gamma}{V} \qquad [T]_\gamma$$

$$[T]_\beta = Q^{-1}[T]_\gamma Q \quad \text{where } \begin{cases} Q = [I]_\beta^\gamma \\ [v]_\gamma = [I]_\beta^\gamma [v]_\beta \end{cases}$$

Proof:

$$\begin{cases} T: \overset{\gamma}{V} \longrightarrow \overset{\gamma}{V} \\ \quad v \longmapsto T(v) \\ \quad [v]_\gamma \longmapsto \underline{[T(v)]_\gamma = [T]_\gamma [v]_\gamma} \quad (*) \end{cases}$$

$$\begin{cases} T: \overset{\beta}{V} \longrightarrow \overset{\beta}{V} \\ \quad v \longmapsto T(v) \\ \quad [v]_\beta \longmapsto \underline{[T(v)]_\beta = [T]_\beta [v]_\beta} \quad (**) \end{cases}$$

$$\left.\underbrace{[v]_\gamma = Q[v]_\beta}_{(*)}\right\} \Rightarrow Q\underbrace{[T(v)]_\beta}_{(**)} = [T]_\gamma Q [v]_\beta \Big\}$$

$$\Rightarrow Q[T]_\beta [v]_\beta = [T]_\gamma Q [v]_\beta, \ \forall v \in V.$$

(Lemma) $\Rightarrow Q[T]_\beta = [T]_\gamma Q \Rightarrow [T]_\beta = Q^{-1}[T]_\gamma Q.$

Def If $A = Q^{-1}BQ$ for some invertible Q, then we say A is similar to B.

② **Def** $T: V \to V$ (for finite dim V) is called diagonalizable if $\exists\ \beta$ s.t. $[T]_\beta$ is diagonal.

Def A matrix $A \in F^{n\times n}$ is digonalizable if L_A is diagonalizable.

1) $L_A : F^n \to F^n$ $\quad\quad \gamma = \{e_1, \cdots, e_n\}$

$$\begin{array}{ccc} x & \to & Ax \\ \parallel & & \parallel \\ [x]_\gamma & & [L_A(x)]_\gamma = [L_A]_\gamma [x]_\gamma = [L_A]_\gamma x \end{array}$$

So $[L_A]_\gamma = A$.

$\beta = \{\underline{v_1, \cdots, v_n}\} \subseteq F^n$

$$[x]_\beta \mapsto [L_A(x)]_\beta = [L_A]_\beta [x]_\beta$$

$$Q = [I]_\beta^\gamma \Rightarrow [L_A]_\beta = Q^{-1} [L_A]_\gamma Q$$
$$= Q^{-1} A Q$$

a) L_A being diagonalizable means $\boxed{Q^{-1}AQ}$ is diagonal

where $Q = [I]_\beta^\gamma$ $\left\{\begin{array}{l}\gamma = \{e_1, \cdots, e_n\} \\ \beta \text{ is the one in def.}\end{array}\right.$

b) A diagonalizable matrix is similar to a diagonal matrix.

c) What is Q? $\quad Q = [v_1\ v_2\ \cdots\ v_n]$

$v_i = [v_i]_\gamma = Q[v_i]_\beta = Q e_i = i\text{-th col of } Q.$

③ Theorem 5.1

(a) $T: V \to V$ is diagonalizable if and only if $\exists$ a basis β consisting of eigenvectors of T.

(b) If T is diagonalizable under β ($[T]_\beta$ is diagonal) $\quad \beta = \{v_1, \cdots, v_n\}$

then v_i are eigenvectors of T and

$$[T]_\beta = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \text{ with } T(v_i) = \lambda_i v_i.$$

Proof: T is diagonalizable $\Leftrightarrow [T]_\beta = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$

for some $\beta = \{v_1, \dots, v_n\}$

$\Leftrightarrow [T(v_i)]_\beta = [T]_\beta [v_i]_\beta = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ d_i \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$\Leftrightarrow T(v_i) = 0 \cdot v_1 + 0 \cdot v_2 + \dots + 0 \cdot v_{i-1} + d_i v_i + 0 \cdot v_{i+1} + \dots$

$= d_i v_i$

Ex: $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in \mathbb{C}^{2\times 2}$ $\quad \det(A - tI) = \left| \begin{pmatrix} \cos\theta - t & -\sin\theta \\ \sin\theta & \cos\theta - t \end{pmatrix} \right|$

$= (\cos\theta - t)^2 + \sin^2\theta$

$= t^2 - 2\cos\theta t + 1 = 0$

$t = \cos\theta \pm i \sin\theta$

$\boxed{L_A : \mathbb{C}^2 \to \mathbb{C}^2 \text{ over } \mathbb{C}}$ is diagonalizable

$\boxed{L_A : \mathbb{R}^2 \to \mathbb{R}^2 \text{ over } \mathbb{R}}$ is NOT diagonalizable

Remark: A matrix A is diagonalizable means:

$\exists\, Q$ s.t. $Q^{-1}AQ$ is diagonal

and cols of Q are eigenvectors
diagonal entries of $Q^{-1}AQ$ are eigenvalues
$\Big\}$ of A

④ Def $f(t) = \det([T]_\beta - tI)$ is called characteristic polynomial of T

$\det([T]_\beta)$ is called det of T.

Theorem ① det and $f(t)$ are the same for T

under different bases.

② det and $f(t)$ are the same for similar matrices

Proof: Only need to prove ② because $[T]_\beta = Q^{-1}[T]_\gamma Q$ where $Q = [I]_\beta^\gamma$.

Lemma: $\det(AB) = \det(BA)$ $A, B, C \in F^{n\times n}$

$\det(ABC) = \det(BCA) = \det(CAB)$

Proof: $\det(AB) = \det(A)\det(B) = \det(BA)$

$\det(ABC) = \det(AB)\det(C) = \det(A)\det(B)\cdot\det(C)$.

If $A = Q^{-1}BQ$, then

$$\det(A) = \det(Q^{-1}BQ) = \det(BQQ^{-1}) = \det(B)$$

$$\det(A - tI) = \det(Q^{-1}BQ - tI) = \det(Q^{-1}BQ - Q^{-1}(tI)Q) = \det[Q^{-1}(B - tI)Q] = \det(B - tI).$$

⑤ Theorem 5.5 $T: V \to V$ $\dim(V) = n$

$\lambda_1, \cdots, \lambda_k$ are distinct eigenvalues

For each λ_i, S_i is an independent set of eigenvectors of λ_i.

then $S = S_1 \cup S_2 \cup \cdots \cup S_k$ is independent.

Proof: By Induction

1) If $k=1$, nothing to prove.

2) Assume it is true for $k-1$. (Induction Hypothesis)

Let $S_i = \{v_1^i, \ldots, v_{n_i}^i\}$ for λ_i.

$$\sum_{i=1}^{k} \sum_{j=1}^{n_i} a_{ij} v_j^i = \vec{0}$$ linear comb of vectors in S.

$a_{ij} \in F$

Apply $(T-\lambda_k I)$ for both sides

$$\sum_{i=1}^{k} \sum_{j=1}^{n_i} a_{ij} [T(v_j^i) - \lambda_k v_j^i] = \vec{0}$$

$$\Rightarrow \sum_{i=1}^{k-1} \sum_{j=1}^{n_i} a_{ij} [\lambda_i v_j^i - \lambda_k v_j^i] = \vec{0}$$

$$\Rightarrow \underbrace{\sum_{i=1}^{k-1} \sum_{j=1}^{n_i}}\, a_{ij} \underbrace{(\lambda_i - \lambda_k)}_{\neq 0} v_j^i = \vec{0}$$

linear comb in union of $(k-1)$ sets

Induction Hypothesis $\Rightarrow a_{ij}(\lambda_i - \lambda_k) = 0$

$\Rightarrow a_{ij}, \quad 1 \le i \le k-1$

$$\Rightarrow \sum_{j=1}^{n_k} a_{kj} v_j^k = \vec{0}$$

S_k is independent $\Rightarrow a_{kj} = 0$.