Chapter 6 (In this Chapter, $F=\mathbb{R}$ or $\mathbb{C}$)

- Inner Product is a generalization of dot product.

__Def__ V is a vector space over F ($\mathbb{R}$ or $\mathbb{C}$).

An inner product on V is a function of two vectors $u, v \in V$ s.t. $\langle u, v\rangle \in F$, $u, v, w \in V$, $c \in F$

(a) $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$

(b) $\langle cu, v\rangle = c\langle u, v\rangle$

(c) $\overline{\langle x, y\rangle} = \langle y, x\rangle$ bar means complex conjugate

(d) $u \neq \vec{0} \Rightarrow \langle u, u\rangle$ is a real positive number.

Remark: ① $\left.\begin{matrix}(a)\\(b)\end{matrix}\right\} \Rightarrow \langle u, v\rangle$ is linear in the first component.

② $\left.\begin{matrix}(b)\\(d)\end{matrix}\right\} \Rightarrow \langle u, u\rangle = 0$ if and only if $u = \vec{0}$

$\langle \vec{0}, \vec{0}\rangle = \langle 0\cdot\vec{0}, \vec{0}\rangle = 0\langle \vec{0}, \vec{0}\rangle = 0.$

$a_i \in F$
$u_i \in V$

③ $\langle \sum_{i=1}^{n} a_i u_i, v\rangle = \sum_{i=1}^{n} a_i \langle u_i, v\rangle$

conjugate linear

$a, b \in \mathbb{C}$

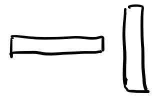
$\overline{ab} = \bar{a}\cdot\bar{b}$

$$\begin{aligned} \langle u, \sum_{i=1}^{n} b_i v_i\rangle &= \overline{\langle \sum_{i=1}^{n} b_i v_i, u\rangle} \\ &= \overline{\sum_{i=1}^{n} b_i \langle v_i, u\rangle} \\ &= \sum_{i=1}^{n} \bar{b}_i \overline{\langle v_i, u\rangle} \\ &= \sum_{i=1}^{n} \bar{b}_i \langle u, v_i\rangle \end{aligned}$$

$F=\mathbb{R} \Rightarrow$ inner prod is also linear in the second component.

Examples: ① $x, y \in \mathbb{C}^n$ $\quad y^* = \bar{y}^T$

Define $\langle x, y\rangle = y^* x = [\bar{y}_1 \cdots \bar{y}_n]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$



$= \sum_{i=1}^{n} x_i \bar{y}_i$ is an inner product.

(Standard Inner Prod)

$x = \begin{pmatrix} 1+i \\ 4 \end{pmatrix}, \; y = \begin{pmatrix} 2-3i \\ 4+5i \end{pmatrix} \in \mathbb{C}^2$

$\langle x, y\rangle = y^* x = [2+3i \quad 4-5i]\begin{bmatrix} 1+i \\ 4 \end{bmatrix}$

$= (2+3i)(1+i) + (4-5i)4 = 15 - 15i$

$x = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \quad y = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

$\langle x, y\rangle = y^* x = y^T x = x \cdot y$

$\underset{a+ib}{\underset{\shortparallel}{x}}, \underset{c+id}{\underset{\shortparallel}{y}} \in \mathbb{C}$

$\langle x, y\rangle = y^* x = \bar{y} x = (c-id)(a+ib)$

$= ac + bd - iad + ibc$

② $x, y \in \mathbb{C}^n$, $\langle x, y\rangle = x^* y$ is NOT an inner prod.

③ $V = C([0,1]) = \{$continuous real-valued functions define on $[0,1]\}$

$F = \mathbb{R}$

$\forall f(t), g(t) \in V$,

Define $\langle f, g\rangle = \int_0^1 f(t) g(t)\, dt$

(a), (b) hold because it is linear in $f(t)$

(c) holds because $\begin{cases} \langle f, g\rangle = \langle g, f\rangle \\ F = \mathbb{R} \end{cases}$

(d) holds because $\langle f,f\rangle = \int_0^1 f^2(t)\,dt > 0$ unless $f(t)\equiv 0$.

④ $H = \{$continuous complex-valued functions defined on $[0,2\pi]\}$.

$$e^{ikt} = \cos(kt) + i\sin(kt)$$

$$e^{ilt} = \cos(lt) + i\sin(lt)$$

$$e^{ikt}\cdot e^{ilt} = e^{i(k+l)t}$$

Trig Formulae $\Rightarrow (\cos(kt)+i\sin(kt))(\cos(lt)+i\sin(lt))$

$$= \cos(k+l)t + i\sin(k+l)t$$

$\forall f(t), g(t) \in H$

Define $\langle f,g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}\,dt$

(d) $\langle f,f\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{f(t)}\,dt = \frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2\,dt$

$$A^* = \overline{A}^T$$

$$A = \begin{pmatrix} i & 1+2i \\ 2 & 3+4i \end{pmatrix} \quad A^* = \begin{pmatrix} -i & 2 \\ 1-2i & 3-4i \end{pmatrix}$$

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⑤ $\forall A, B \in V \in F^{n\times n}$, define $\langle A,B\rangle = tr(B^*A)$

Frobenius Inner Product

$$tr(A) = \sum_{i=1}^n A_{ii}$$

$U, V \in F^{n\times n}$

$$(UV)_{ij} = \sum_{k=1}^n U_{ik}V_{kj}$$

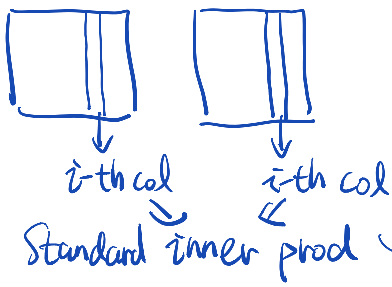
$$tr(B^*A) = tr(\overline{B}^T A)$$

$$= \sum_{i=1}^n (\overline{B}^T A)_{ii}$$

$$= \sum_{i=1}^n \left(\sum_{k=1}^n (\overline{B}^T)_{ik} A_{ki} \right)$$

$$= \sum_{i=1}^n \left(\sum_{k=1}^n \overline{B}_{ki} A_{ki} \right) = \langle vec(A), vec(B)\rangle$$

A B



For $A \in F^{n \times n}$, let $vec(A)$ be a col vec of size n^2 obtained by rearranging cols of A.

Ex: $A \in \mathbb{C}^{3 \times 3}$, $vec(A) = \begin{bmatrix} \; \\ \; \\ \; \end{bmatrix}$ → 1st col of A, → 2nd col of A, → 3rd col of A

(a) $\langle A+B, C \rangle = tr(C^*(A+B))$

$= tr(C^*A + C^*B)$

$= tr(C^*A) + tr(C^*B)$

$= \langle A, C \rangle + \langle B, C \rangle$

(b) $a \in F$, $\langle aA, B \rangle = tr(B^*(aA))$

$= tr(a(B^*A))$

$= a\, tr(B^*A)$

(c) $\overline{\langle A, B \rangle} = \overline{tr(B^*A)} = tr(\overline{B^*A})$

$= tr(\overline{B^*}\,\overline{A})$

$= tr(\overline{\overline{B}^T}\,\overline{A})$

$= tr(B^T \overline{A})$

HW#4 $tr(A) = tr(A^T)$ $= tr[(B^T\overline{A})^T]$

$= tr(\overline{A}^T B) = \langle B, A \rangle$

(d) $\langle A, B\rangle = tr(B^* A) = \sum_{i=1}^{n}\sum_{k=1}^{n} \overline{B_{ki}} A_{ki}$

$A \neq O_{n\times n} \Rightarrow \langle A, A\rangle = \sum_{i=1}^{n}\sum_{k=1}^{n} \overline{A_{ki}} A_{ki}$

$= \sum_{i=1}^{n}\sum_{k=1}^{n} |A_{ki}|^2 > 0$

<u>Def</u> A vector space V over $F = \mathbb{R}$ (or $\mathbb{C}$) with an inner product is called an inner product space.

If V is an inner prod space, W is a subspace of V then W is also an inner prod space with the same inner product.

Theorem 6.1 $x, y, z \in V$

① $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$

② $\langle x, cy\rangle = \bar{c}\langle x, y\rangle$

③ $\langle x, \vec{0}\rangle = \langle \vec{0}, x\rangle = 0$

④ $\langle x, x\rangle = 0 \Leftrightarrow x = \vec{0}$

⑤ $\langle x, y\rangle = \langle x, z\rangle, \ \forall x \in V \Rightarrow y = z$

<u>Proof of ⑤</u>: $\langle x, y\rangle = \langle x, z\rangle$

$\Rightarrow \langle x, y\rangle - \langle x, z\rangle = 0$

($c = -1$ in ②) $\Rightarrow \langle x, y-z\rangle = 0$

Set $x = y - z$,

$\Rightarrow \langle y-z, y-z\rangle = 0$

④ $\Rightarrow y - z = \vec{0} \Rightarrow y = z$.

Def $\|x\| = \sqrt{\langle x, x\rangle}$ is called norm or length of $x \in V$.

Ex: Standard inner prod

$$\Rightarrow x \in F^n, \quad \|x\| = \sqrt{\langle x, x\rangle} = \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

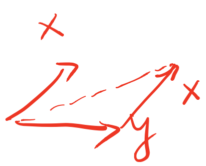
Euclidean Length

Theorem 6.2 $x \in V$, $c \in F$

① $\|cx\| = |c| \cdot \|x\|$

② $\|x\| = 0 \Leftrightarrow x = \vec{0}$

③ $|\langle x, y\rangle| \le \|x\| \cdot \|y\|$ Cauchy-Schartz Inequality



④ $\|x+y\| \le \|x\| + \|y\|$ Triangle Inequality

Proof of ③: If $y = \vec{0}$, $\begin{cases} RHS = 0 \\ LHS = 0 \end{cases}$

If $y \ne \vec{0}$, $0 \le \langle x - cy, x - cy\rangle$

$$= \underline{\langle x, x - cy\rangle} - c\underline{\langle y, x - cy\rangle}$$

$$= \langle x, x\rangle - \bar{c}\langle x, y\rangle - c\langle y, x\rangle + c\bar{c}\langle y, y\rangle$$

Set $c = \dfrac{\langle x, y\rangle}{\langle y, y\rangle} \Rightarrow \bar{c}\langle x, y\rangle = \dfrac{\overline{\langle x, y\rangle}}{\overline{\langle y, y\rangle}}\langle x, y\rangle$

$$= \frac{\langle y, x\rangle}{\langle y, y\rangle}\langle x, y\rangle$$

$$c\langle y, x\rangle = \frac{\langle x, y\rangle}{\langle y, y\rangle}\langle y, x\rangle$$

$$c\bar{c}\langle y, y\rangle = \frac{\langle x, y\rangle}{\langle y, y\rangle}\underline{\frac{\langle y, x\rangle}{\langle y, y\rangle}}\langle y, y\rangle = \frac{\langle x, y\rangle}{\langle y, y\rangle}\langle y, x\rangle$$

$$\Rightarrow \quad 0 \leq \langle x - cy, x - cy \rangle$$

$$= \langle x, x \rangle - \frac{\langle x, y \rangle}{\langle y, y \rangle} \langle y, x \rangle$$

$$\Rightarrow \quad \frac{\langle x, y \rangle}{\langle y, y \rangle} \langle y, x \rangle \leq \langle x, x \rangle$$

$$\Rightarrow \quad \langle x, y \rangle \langle y, x \rangle \leq \langle x, x \rangle \langle y, y \rangle$$

$$\Rightarrow \quad \langle x, y \rangle \overline{\langle x, y \rangle} \leq \langle x, x \rangle \langle y, y \rangle$$

$$\Rightarrow \quad |\langle x, y \rangle|^2 \leq \|x\|^2 \|y\|^2$$

④ $$\|x+y\|^2 = \langle x+y, x+y \rangle$$

$$= \langle x, x \rangle + \underbrace{\langle y, x \rangle + \langle x, y \rangle} + \langle y, y \rangle$$

$$= \|x\|^2 + \|y\|^2 + \underbrace{2\mathrm{Re}(\langle x, y \rangle)}_{\substack{\leq 2|\langle x, y \rangle| \\ \leq 2\|x\| \cdot \|y\|}}$$

$$\leq \|x\|^2 + \|y\|^2 + 2\|x\| \cdot \|y\|$$

$$= (\|x\| + \|y\|)^2$$