Review of 6.1, 6.2

- An Inner Product Space is a V.S. with $\langle , \rangle$ defined. over $F = \mathbb{R}$ (or $\mathbb{C}$).
- Length/Norm of a vector : $\|v\| = \sqrt{\langle v, v\rangle}$
- $\{v_1, \cdots, v_n\}$ orthonormal if $\langle v_i, v_j\rangle = \delta_{ij} = \begin{cases} 1 & , i=j \\ 0 & , i\neq j \end{cases}$
- Gram-Schmidt process
- If $\beta = \{v_1, \cdots, v_n\}$ is an orthonormal basis, then $v = \sum_{i=1}^{n} \langle v, v_i \rangle v_i$
- If $\beta = \{v_1, \cdots, v_n\}$ is an orthogonal basis, then

$$v = \sum_{i=1}^{n} \langle v, \frac{v_i}{\|v_i\|}\rangle \frac{v_i}{\|v_i\|}$$
$$= \sum_{i=1}^{n} \frac{\langle v, v_i\rangle}{\|v_i\|^2} v_i$$

- $S \subseteq V$ is a subset

$S^\perp = \{x \in V : \langle x, y\rangle = 0, \forall y \in S\}$

Theorem 6.6 W is a finite-dim subspace of V

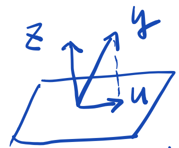
$\forall y \in V$, there exist unique $u \in W$, $z \in W^\perp$ s.t.

$$y = u + z$$

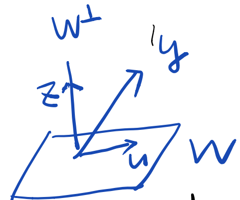
If $\{v_1, \cdots, v_k\}$ is orthonormal basis of W,

$$u = \sum_{i=1}^{k} \langle y, v_i\rangle v_i .$$

Elementary Case : $W = \{xy\text{-plane}\}$



Abstract Case:



Proof: Define $u = \sum_{i=1}^{k} \langle y, v_i \rangle v_i \in W$

Define $z = y - u$, then $u + z = y$.

Need to show $z \in W^{\perp}$ and uniqueness.

$\forall x \in W$, $x = \sum_{j=1}^{k} \langle x, v_j \rangle v_j$

$$\langle x, z \rangle = \left\langle \sum_{j=1}^{k} \langle x, v_j \rangle v_j , z \right\rangle$$

$$= \sum_{j=1}^{k} \langle x, v_j \rangle \langle v_j, z \rangle$$

$$= \sum_{j=1}^{k} \langle x, v_j \rangle \langle v_j, y - u \rangle$$

$$= \sum_{j=1}^{k} \langle x, v_j \rangle \left\langle v_j, y - \sum_{i=1}^{k} \langle y, v_i \rangle v_i \right\rangle$$

$$= \sum_{j=1}^{k} \langle x, v_j \rangle \left(\langle v_j, y \rangle - \left\langle v_j, \sum_{i=1}^{k} \langle y, v_i \rangle v_i \right\rangle \right)$$

$$= \sum_{j=1}^{k} \langle x, v_j \rangle \left(\langle v_j, y \rangle - \sum_{i=1}^{k} \overline{\langle y, v_i \rangle} \underbrace{\langle v_j, v_i \rangle}_{\delta_{ij}} \right)$$

$$= \sum_{j=1}^{k} \langle x, v_j \rangle \left(\langle v_j, y \rangle - \sum_{i=1}^{k} \overline{\langle y, v_i \rangle} \delta_{ij} \right)$$

$$= \sum_{j=1}^{k} \langle x, v_j \rangle \left(\langle v_j, y \rangle - \overline{\langle y, v_j \rangle} \right)$$

$$= \sum_{j=1}^{k} \langle x, v_j \rangle \left(\langle v_j, y \rangle - \langle v_j, y \rangle \right)$$

$$= 0.$$

Assume $\exists\, u' \in W,\ z' \in W^\perp$ s.t. $y = u' + z'$.

Then $u + z = y = u' + z'$.

$$\Rightarrow u - u' = z' - z = v$$

$$\left.\begin{array}{l} u \in W \\ u' \in W \end{array}\right\} \Rightarrow v = u - u' \in W$$

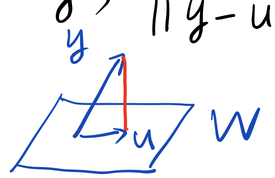
$$\left.\begin{array}{l} z' \in W^\perp \\ z \in W^\perp \end{array}\right\} \Rightarrow v = z' - z \in W^\perp$$

$$\left.\begin{array}{l} v \in W \\ v \in W^\perp \end{array}\right\} \Rightarrow \langle v, v \rangle = 0 \Rightarrow v = \vec{0}$$

$$\Rightarrow \begin{cases} u = u' \\ z = z' \end{cases}.$$

<u>Corollary</u>: In Theorem 6.6, u is the vector in W that is "closest" to y.

Namely, $\|y - u\| \le \|y - x\|$, $\forall x \in W$



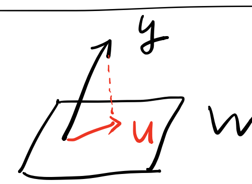
Proof: $\quad u = \sum_{i=1}^{k} \langle y, v_i \rangle v_i$

$$\forall x \in W, \quad \|y-x\|^2 = \|(u+z)-x\|^2$$
$$= \|(u-x)+z\|^2$$
$$= \langle (u-x)+z, (u-x)+z \rangle$$
$$= \|u-x\|^2 + \|z\|^2 + \langle u-x, z\rangle + \langle z, u-x \rangle$$

$$\left.\begin{array}{l} \left.\begin{array}{l} u \in W \\ x \in W \end{array}\right\} \Rightarrow u-x \in W \\ \qquad\qquad z \in W^{\perp} \end{array}\right\} \Rightarrow \langle u-x, z\rangle = 0$$

$$= \|u-x\|^2 + \|z\|^2$$
$$\geq \|z\|^2$$
$$= \|y-u\|^2$$

Projection of y onto W:



is $\begin{cases} \sum_{i=1}^{k} \langle y, v_i \rangle v_i & \text{if } \{v_1, \dots, v_k\} \text{ is orthonormal basis.} \\ \sum_{i=1}^{k} \langle y, \frac{v_i}{\|v_i\|} \rangle \frac{v_i}{\|v_i\|} & \text{if } \{v_1, \dots, v_k\} \text{ is orthogonal basis} \\ = \sum_{i=1}^{k} \frac{\langle y, v_i \rangle}{\|v_i\|^2} v_i & \end{cases}$

Gram-Schmidt for $\{w_1, w_2, w_3\}$

$$v_1 = w_1$$
$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1$$

$$v_3 = w_3 - \frac{\langle w_3, v_1\rangle}{\|v_1\|^2} v_1 - \underbrace{\frac{\langle w_3, v_2\rangle}{\|v_2\|^2} v_2}_{\text{Projection of } w_3 \text{ onto Span}\{v_2\}}$$

Theorem 6.7 $S = \{v_1, \ldots, v_k\}$ is orthonormal in an n-dim prod space V $(k \le n)$

Then 1) S can be extended to an orthonormal basis $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$

2) If $W = \text{Span}(S)$, then $S_1 = \{v_{k+1}, \ldots, v_n\}$ is an orthonormal basis for $W^\perp$.

3) If W is any subspace of V,

$$\dim(V) = \dim(W) + \dim(W^\perp).$$

Sketchy Proof: 1) Replacement Theorem

$\Rightarrow$ S can be extended to a basis $\{v_1, \ldots, v_k, w_{k+1}, \ldots, w_k\}$

Gram-Schmidt

2) $S_1 \subseteq W^\perp$ and S_1 is independent

Only need to show $W^\perp \subseteq \text{Span}(S_1)$

$\forall x \in W^\perp$, $x \in V \Rightarrow x = \sum_{i=1}^{n} \langle x, v_i\rangle v_i$

$x \in W^{\perp} \Rightarrow \langle x, v_i \rangle = 0, \quad i = 1, \cdots, k$

$$\Rightarrow x = \sum_{i=k+1}^{n} \langle x, v_i \rangle v_i \in \mathrm{Span}(S_1)$$

3) Let $\{u_1, \cdots, u_m\}$ be orthonormal basis of W

Apply 1) and 2).

6.3 Adjoint Operator

$$A^* = \overline{A}^T$$

Q: if $[T]_\beta = A$, what is the operater s.t. the matrix representation under β is A^*?

Theorem 6.8 V is finite-dim prod space over F

If $g: V \to F$ is linear, $x \mapsto g(x)$

there exists unique $y \in V$ s.t. $g(x) = \langle x, y \rangle$, $\forall x \in V$.

Ex: $g: \mathbb{R}^2 \to \mathbb{R}$, $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto x$ $\quad g\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \left\langle \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle$

Proof: $\beta = \{v_1, \cdots, v_n\}$ is orthonormal basis of V.

Define $y = \sum_{i=1}^{n} \overline{g(v_i)}\, v_i \in V$

Define $h: V \longrightarrow F$, $x \longmapsto \langle x, y \rangle$ is linear

Want to show $h(x) = g(x)$.

$$h(v_j) = \langle v_j, y \rangle = \langle v_j, \sum_{i=1}^{n} \overline{g(v_i)} v_i \rangle$$

$$= \sum_{i=1}^{n} g(v_i) \underbrace{\langle v_j, v_i \rangle}_{\delta_{ij}}$$

$$= \sum_{i=1}^{n} g(v_i) \delta_{ij}$$

$$= g(v_j)$$

$\forall x \in V$, $x = \sum_{i=1}^{n} \langle x, v_i \rangle v_i$

$$h(x) = h\left(\sum_{i=1}^{n} \langle x, v_i \rangle v_i\right)$$

$$= \sum_{i=1}^{n} \langle x, v_i \rangle h(v_i)$$

$$= \sum_{i=1}^{n} \langle x, v_i \rangle g(v_i)$$

$$= g\left(\sum_{i=1}^{n} \langle x, v_i \rangle v_i\right)$$

$$= g(x).$$

Theorem 6.9 $T : V \to V$ T^* is called adjoint of T.

there exists unique $T^* : V \to V$ s.t.

$$* \quad \boxed{\langle T(x), y \rangle = \langle x, T^*(y) \rangle, \quad \forall x, y \in V.}$$

Remark: HW#7 P4, $\langle \underset{L_A(x)}{Ax}, y \rangle = \langle x, \underset{L_{A^*}(y)}{A^* y} \rangle$

standard inner prod for $\mathbb{C}^n$.

$(L_A)^* = L_{A^*}$

Proof: For a $y \in V$,

define $g: V \longrightarrow F$

$x \longmapsto \langle T(x), y \rangle$

We can verify g is linear

Theorem 6.8 $\Rightarrow \exists\, y' \in V$ s.t.

$g(x) = \langle x, y' \rangle, \ \forall x \in V.$

Define $T^*: V \longrightarrow V$

$y \longmapsto y'$

Verify T^* is linear.

$$\underline{\langle T(x), y \rangle} = g(x) = \langle x, y' \rangle = \underline{\langle x, T^*(y) \rangle}.$$

Theorem 6.10 $\quad [T^*]_\beta = [T]_\beta^*$,

where $\beta = \{v_1, \ldots, v_n\}$ is orthonormal basis.

Proof: $A = [T]_\beta$

$B = [T^*]_\beta$

Want to show $B = A^*$.

$$T: \overset{\beta}{V} \longrightarrow \overset{\beta}{V}$$

$$A_{ij} = \langle T(v_j), v_i \rangle$$

$$\begin{cases} T[v_1] = \langle T(v_1), v_1\rangle v_1 + \langle T(v_1), v_2\rangle v_2 + \cdots + \langle T(v_1), v_n\rangle v_n \\ T[v_2] \\ \vdots \\ T[v_n] \end{cases}$$

$$A_{ij} = \langle T(v_j), v_i\rangle$$

$$T^*: \overset{\beta}{V} \longrightarrow \overset{\beta}{V}$$

$$\begin{aligned} B_{ij} &= \langle T^*(v_j), v_i\rangle \\ &= \overline{\langle v_i, T^*(v_j)\rangle} \\ &= \overline{\langle T(v_i), v_j\rangle} \\ &= \overline{A_{ji}} \end{aligned}$$

$$\Rightarrow B = \bar{A}^T = A^*.$$