Theorem 6.18 $\dim(V) = n$

The following are equivalent:

(a) $T^*T = I$ $\beta \Rightarrow [T^*T]_\beta = [T^*]_\beta [T]_\beta = [T]^*_\beta [T]_\beta$

(b) $TT^* = I$ Recall dot product: $u \cdot v = \cos\theta \, \|u\| \cdot \|v\|$

(c) $\langle T(x), T(y) \rangle = \langle x, y \rangle$, $\forall x, y \in V$.

(d) If β is an orthonormal basis

then $T(\beta)$ — — —— — — .

(e) $\exists$ an orthonormal basis s.t.

$T(\beta)$ — — — —— —

(f) $\|T(x)\| = \|x\|$, $\forall x \in V$.

Proof: (a) $\Leftrightarrow$ (b) by matrix representation.

(a) $\Rightarrow$ (c): $\langle x, y \rangle = \langle T^*T(x), y \rangle$

$= \langle T(x), T(y) \rangle$

(c) $\Rightarrow$ (d): $\beta = \{v_1, \dots, v_n\}$ is orthonormal basis

$\underline{\langle T(v_i), T(v_j) \rangle} = \langle v_i, v_j \rangle = \underline{\delta_{ij}} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

$\underline{\Rightarrow \{T(v_1), \dots, T(v_n)\} \text{ is orthonormal}}$

Orthonormal set is independent.

$\Rightarrow T(\beta)$ is orthonormal basis.

(d) $\Rightarrow$ (e): obvious

(e) $\Rightarrow$ (f): Let $\beta = \{v_1, \dots, v_n\}$ be that orthonormal basis.

$$\forall x \in V, \quad x = \sum_{i=1}^{n} a_i v_i$$

$$\begin{aligned}
\|x\|^2 &= \langle x, x \rangle \\
&= \Big\langle \sum_{i=1}^{n} a_i v_i, \sum_{j=1}^{n} a_j v_j \Big\rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \overline{a_j} \, \underline{\langle v_i, v_j \rangle} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \overline{a_j} \, \underline{\delta_{ij}} \\
&= \sum_{i=1}^{n} |a_i|^2
\end{aligned}$$

$$\begin{aligned}
\|T(x)\| &= \langle T(x), T(x) \rangle \\
&= \Big\langle T\Big(\sum_{i=1}^{n} a_i v_i\Big), T\Big(\sum_{j=1}^{n} a_j v_j\Big) \Big\rangle \\
&= \Big\langle \sum_{i=1}^{n} a_i T(v_i), \sum_{j=1}^{n} a_j T(v_j) \Big\rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \overline{a_j} \, \underline{\langle T(v_i), T(v_j) \rangle} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \overline{a_j} \, \delta_{ij} = \sum_{i=1}^{n} |a_i|^2
\end{aligned}$$

$$\Rightarrow \|T(x)\|^2 = \|x\|^2$$

(f) $\Rightarrow$ (a): Lemma $\left.\begin{array}{l} U \text{ is self-adjoint} \\ \langle x, U(x) \rangle = 0, \ \forall x \in V \end{array}\right\} \Rightarrow U(x) = \vec{0}, \ \forall x \in V.$

Proof: $0 = \langle \underbrace{x + U(x)}_{y}, \underbrace{U(x + U(x))}_{U(y)} \rangle$

$$= \langle x + U(x),\ U(x) + U^2(x) \rangle$$

$$= \underbrace{\langle x, U(x) \rangle}_{= 0} + \langle x, U^2(x) \rangle + \langle U(x), U(x) \rangle + \langle \underbrace{U(x)}_{z}, \underbrace{U^2(x)}_{U(z)} \rangle$$

$$= \langle x, U^2(x) \rangle + \langle U(x), U(x) \rangle$$

$$= \langle U^*(x), U(x) \rangle + \langle U(x), U(x) \rangle$$

$$= \langle U(x), U(x) \rangle + \langle U(x), U(x) \rangle$$

$$= 2\|U(x)\|^2 \Rightarrow \|U(x)\| = 0 \Rightarrow U(x) = \vec{0},\ \forall x \in V$$

$$\langle x, x \rangle = \|x\|^2 = \|T(x)\|^2 = \langle T(x), T(x) \rangle = \langle x, T^*T(x) \rangle$$

$$\Rightarrow \quad \langle x, x \rangle - \langle x, T^*T(x) \rangle = 0$$

$$\Rightarrow \quad \langle x, x - T^*T(x) \rangle = 0$$

$$\Rightarrow \quad \langle x, \underbrace{(I - T^*T)}_{U}(x) \rangle = 0$$

$$\langle x, U(x) \rangle = 0, \quad \forall x \in V$$

$$U^* = (I - T^*T)^* = I^* - (T^*T)^* = I - T^*(T^*)^* = I - T^*T = U$$

$$\text{Lemma} \Rightarrow U(x) = \vec{0}, \ \forall x \in V$$

$$\Rightarrow (I - T^*T)(x) = \vec{0}, \ \forall x \in V \Rightarrow T^*T = I.$$

Corollary 1. If $T: V \to V$, $\dim(V) = n$, $F = \mathbb{R}$, then

T has n orthonormal eigenvectors with $|\lambda_i| = 1$ $\quad \lambda_i = \pm 1$

if and only if T is $\underbrace{\text{self-adjoint}}_{T^* = T}$ and orthogonal.

Proof: ONLY IF Assume $\begin{cases} T(v_i) = \lambda_i v_i \\ |\lambda_i| = 1 \\ \langle v_i, v_j \rangle = 1 \end{cases}$ $\forall i, j \in \{1, \dots, n\}$.

$$\Rightarrow \langle T(v_i), T(v_j) \rangle = \langle \lambda_i v_i, \lambda_j v_j \rangle$$
$$= \lambda_i \overline{\lambda_j} \langle v_i, v_j \rangle$$
$$= \begin{cases} \lambda_i \overline{\lambda_i} \cdot 1 = |\lambda_i|^2 = 1, & \text{if } i = j. \\ 0, & \text{if } i \neq j \end{cases}$$
$$= \delta_{ij}$$

$\Rightarrow \{T(v_1), \dots, T(v_n)\}$ is orthonormal basis

(e) in Theorem 6.18 $\Rightarrow$ T is orthogonal.

Theorem 6.17 $\Rightarrow$ T is self-adjoint.

IF : Assume T is self-adjoint and orthogonal.

Theorem 6.17, self-adjoint $\Rightarrow$ T has n orthonormal eigenvectors v_i

$$\Rightarrow \begin{cases} T(v_i) = \lambda_i v_i \\ \langle v_i, v_j \rangle = \delta_{ij} \end{cases}$$

T is orthogonal $\Rightarrow |\lambda_i| \cdot \|v_i\| = \|\lambda_i v_i\| = \|T(v_i)\| = \|v_i\|$

$$\Rightarrow |\lambda_i| = 1$$

Corollary 2: $T: V \to V$, $\dim(V) = n$, $F = \mathbb{C}$

T has n orthonormal eigenvectors with $|\lambda_i| = 1$ if and only if T is unitary.

Proof: HW.

<u>Def</u> $A \in \mathbb{R}^{n\times n}$ is orthogonal if $A^TA = AA^T = I$.

$A \in \mathbb{C}^{n\times n}$ is unitary if $A^*A = AA^* = I$.

real: $$A^TA = \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix} \overbrace{\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}}^{A} = \boxed{B}$$

$$B_{ij} = v_i^T v_j = \langle v_j, v_i \rangle$$

$$B = I \iff \langle v_j, v_i \rangle = \delta_{ij}$$

Orthogonal matrix has orthonormal cols/rows.

$$AA^T = \begin{bmatrix} u_1^T \\ \vdots \\ u_n^T \end{bmatrix} \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} = \boxed{C}$$

$$C_{ij} = u_i^T u_j = \langle u_j, u_i \rangle$$

$C = I \iff \{u_1, \dots, u_j\}$ are orthonormal.

Complex Case:

$$A^*A = \begin{bmatrix} v_1^* \\ \vdots \\ v_n^* \end{bmatrix} \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} = \boxed{B}$$

$$B_{ij} = v_i^* v_j = \langle v_j, v_i \rangle$$

$$B = I \iff \langle v_j, v_i \rangle = \delta_{ij}$$

Unitary matrices have orthonormal cols/rows.

Example: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is orthogonal.

Theorem Complex normal matrix $A \in \mathbb{C}^{n\times n}$ has n orthonormal eigenvectors in $\mathbb{C}^n$.

Proof: Apply Theorem 6.16 to $L_A : \mathbb{C}^n \to \mathbb{C}^n$
$$x \mapsto Ax$$

$$\begin{cases} A \text{ is normal} \Rightarrow AA^* = A^*A \\ (L_A)^* = L_{A^*} \text{ because } \langle L_A(x), y\rangle = \langle Ax, y\rangle = \langle x, A^*y\rangle \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad = \langle x, L_{A^*}(y)\rangle \end{cases}$$

$$\Rightarrow (L_A)^* L_A = L_{A^*} L_A = L_{A^*A} = L_{AA^*} = L_A L_{A^*}$$
$$= L_A (L_A)^*$$

$\Rightarrow L_A$ is normal

Theorem 6.16 $\Rightarrow$ L_A has n orthonormal eigenvectors

$\Rightarrow A$ — — — — ——————

$\boxed{\text{Fact}}$ Assume $A \in \mathbb{C}^{n\times n}$ has n orthonormal eigenvectors v_i

$$Q = [v_1 \cdots v_n] \in \mathbb{C}^{n\times n}$$

$$AQ = [Av_1 \cdots Av_n]$$
$$= [\underline{\lambda_1 v_1} \cdots \lambda_n v_n]$$
$$= [v_1 \cdots v_n]\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$
$$= QD$$

$$\Rightarrow \quad Q^{-1}AQ = D$$ A is similar to D.

$\langle v_i, v_j \rangle = \delta_{ij} \Rightarrow Q$ has orthonormal cols

$$\Rightarrow Q^*Q = I$$
$$\Rightarrow Q^{-1} = Q^*.$$

$$\Rightarrow \quad Q^*AQ = D$$

Def If $Q^*AQ = B$ for a unitary Q (or orthogonal Q)

then we say A and B are unitarily equivalent.

(orthogonally equivalent)

Theorem 6.19 $\quad A \in \mathbb{C}^{n\times n}$

A is normal $\Leftrightarrow$ A is unitarily equivalent to a diagonal matrix.

$AA^* = A^*A$

Proof: "$\Rightarrow$" Theorem + Fact

"$\Leftarrow$" Assume $A = P^*DP$, $\quad P^*P = PP^* = I$.

$$AA^* = P^*DP(P^*DP)^*$$
$$= P^*DP(P^*D^*(P^*)^*)$$
$$= P^*D\underbrace{PP^*}_{I}D^*P$$
$$= P^*\underline{DD^*}P$$

$$= P^* D^* D P$$

$$A^*A = (P^* D P)^* (P^* D P)$$
$$= P^* D^* \underbrace{P P^*}_{I} D P = P^* D^* D P = AA^*.$$

Combine Theorem 6.16 and 6.19, then the following are equivalent for $A \in \mathbb{C}^{n\times n}$

① $AA^* = A^*A$

② A has n orthonormal eigenvectors.

③ $A = Q^* \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} Q$, $Q^*Q = QQ^* = I$

and cols of Q are eigenvectors

λ_i are eigenvalues.