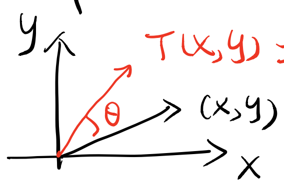
Linear Transformation $T: V \longrightarrow W$

$$v \longmapsto T(v)$$

T preserves two operations
① $T(x+y) = T(x) + T(y)$
② $T(cx) = cT(x)$, $c \in F$

Examples of Linear Transformation from $\mathbb{R}^2 \to \mathbb{R}^2$

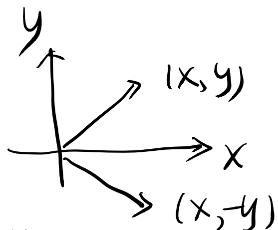
① Rotation



$T(x,y) = (\cos\theta\, x - \sin\theta\, y,\ \sin\theta\, x + \cos\theta\, y)$

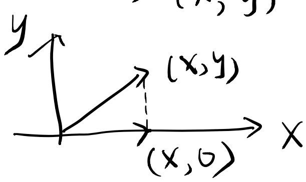
$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

②



Reflection $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

③



Projection $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Ex 1: $V = C(\mathbb{R}) = \{\text{all continuous functions}\}$

$T: V \longrightarrow \mathbb{R}$

$f(x) \longmapsto \int_a^b f(t)\,dt$

Ex 2: $V = \{\text{all differentiable functions}\}$

$W = \{\text{all functions}\}$

$T: V \longrightarrow W$

$f(x) \longmapsto f'(x)$

Def $T: V \to W$

$N(T) = \{x \in V : T(x) = \vec{0}_W\} \subseteq V$ Null Space of T.

$R(T) = \{T(x) : x \in V\} \subseteq W$ Range/Image of T.

Ex: $T: \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \end{bmatrix}$$

$N(T) = \left\{ \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} : z \in \mathbb{R} \right\}$

$R(T) = \mathbb{R}^2$.

$\text{Nullity}(T) = \dim(N(T)) = 1$

$\text{Rank}(T) = \dim(R(T)) = 2$

Theorem 2.1 ① $N(T)$ is a subspace of V

② $R(T)$ is a subspace of W.

Proof: ① T is linear $\Rightarrow T(\vec{0}_V) = \vec{0}_W$

$\Rightarrow \vec{0}_V \in N(T)$

$\forall x, y \in N(T)$, $T(x) = T(y) = \vec{0}_W$

$\forall a, b \in F$, $T(ax+by) = aT(x) + bT(y)$

$= a\vec{0}_W + b\vec{0}_W$

$= \vec{0}_W$

$\Rightarrow ax+by \in N(T)$

$\Rightarrow N(T)$ is closed under $+$ and $\cdot$.

$\Rightarrow N(T)$ is a subspace.

② is similar.

Theorem 2.2 $T: V \to W$

$\beta = \{v_1, \dots, v_n\}$ is a basis of V.

Then $R(T) = \text{Span}(T(\beta)) = \text{Span}(T(v_1), \dots, T(v_n))$

Proof: ① $\forall w \in R(T)$, $w = T(v)$ for some $v \in V$.

$v = a_1 v_1 + \dots + a_n v_n$

$w = T(v) = aT(v_1) + \dots + a_n T(v_n) \in \text{Span}(T(\beta))$

$\Rightarrow R(T) \subseteq \text{Span}(T(\beta))$

② $T(v_i) \in R(T)$, $R(T)$ is a subspace $\Big\} \Rightarrow \text{Span}(T(\beta)) \subseteq R(T)$

Theorem Any subspace containing S also contains $\text{Span}(S)$.

Theorem 2.3 (Dim Thm)

$T: V \to W$ $\quad N(T) \subseteq V \quad R(T) \subseteq W$

V is finite dim $\Rightarrow \underbrace{\dim(N(T))}_{\text{Nullity of } T} + \underbrace{\dim(R(T))}_{\text{rank of } T} = \dim(V)$

Proof: Let $\dim(V) = n$, $N(T) \subseteq V \Rightarrow \dim(N(T)) = k \leq n$.

Let $\{v_1, \cdots, v_k\}$ be basis of $N(T)$.

Replacement Theorem, $\{v_1, \cdots, v_k\}$ is independent $\Big\} \Rightarrow \{v_1, \cdots, v_k\}$ can be extended to a basis of V

$$\beta = \{v_1, \cdots, v_k, v_{k+1}, \cdots, v_n\}$$

Claim $S = \{T(v_{k+1}), \cdots, T(v_n)\}$ is a basis of $R(T)$.

① Theorem 2.2 $\Rightarrow R(T) = \text{Span}\{T(v_1), \cdots, T(v_n)\}$

$$\underset{(T(v_1) = \cdots = T(v_k) = \vec{0}_k)}{=} \text{Span}\{T(v_{k+1}), \cdots, T(v_n)\}$$

$$= \text{Span}(S)$$

② Want to S is independent.

$$b_{k+1} T(v_{k+1}) + \cdots + b_n T(v_n) = \vec{0}_W$$

$$\Rightarrow T(b_{k+1} v_{k+1} + \cdots + b_n v_n) = \vec{0}_W$$

$$\Rightarrow b_{k+1} v_{k+1} + \cdots + b_n v_n \in N(T)$$

$$\Rightarrow b_{k+1} v_{k+1} + \cdots + b_n v_n = a_1 v_1 + \cdots + a_k v_k$$

$\Rightarrow -a_1v_1 - \cdots - a_kv_k + b_{k+1}v_{k+1} + \cdots + b_nv_n = \vec{0}_V$

$\{v_1, \cdots, v_n\}$ is a basis of V

$\Rightarrow$ All coefs are 0.

$\Rightarrow b_i = 0, \forall i.$

$\Rightarrow S$ is independent.

Theorem 2.4 $\quad T: V \to W$

T is one-to-one iff $N(T) = \{\vec{0}_V\}$.

Proof: "IF" Assume $T(x) = T(y)$

$\Rightarrow T(x) - T(y) = \vec{0}_W$

$\Rightarrow T(x-y) = \vec{0}_W$

$\Rightarrow x-y \in N(T)$

$\Rightarrow x-y = \vec{0}_V$

$\Rightarrow x = y.$

"ONLY IF" $\forall z \in N(T)$, $T(z) = \vec{0}_W = T(\vec{0}_V)$

$\Rightarrow z = \vec{0}_V.$

Theorem 2.5 $\quad T: V \to W$ and $\underline{\dim(V) = \dim(W)}$ is finite.

The following are equivalent

① T is one-to-one (injective)

② T is onto (surjective)

③ $\underbrace{\mathrm{rank}(T)}_{\dim(R(T))} = \dim(V)$ $\quad$ ($\mathrm{Nullity}(T) + \mathrm{Rank}(T) = \dim(V)$)

$\Updownarrow$

$\mathrm{Nullity} = 0$

$\Updownarrow$

$N(T) = \{\vec{0}_V\}$

Proof: Theorem 2.4 gives ① $\Leftrightarrow$ ③

② $\Leftrightarrow$ ③ : $\text{rank}(T) = \dim(V) \Leftrightarrow \dim(R(T)) = \dim(W)$

$\Leftrightarrow R(T) = W$

Theorem 2.6 V. W are two V.S.

V has a basis $\{v_1, \cdots, v_n\}$

For any distinct $w_1, \cdots, w_n \in W$,

there is a unique T s.t.

$T(v_i) = w_i, \ \forall i.$

Corollary : $T_1, T_2 : V \longrightarrow W$ are linear

V has a basis $\{v_1, \cdots, v_n\}$

$T_1(v_i) = T_2(v_i), \forall i \Rightarrow T_1 = T_2.$

Proof: $\forall v \in V$, $v = a_1 v_1 + \cdots + a_n v_n$

$T_1(v) = a_1 T(v_1) + \cdots + a_n T(v_n).$

In Span/Generate/Linear Combination, only finite sum is used in the definition.

Ex: $P(\mathbb{R})$ $\beta = \{1, x, x^2, \cdots, x^n, \cdots\}$

$P(\mathbb{R}) = \text{Span}(\beta)$ means that $\forall f(x) \in P(\mathbb{R})$,

there exists finite vectors in β s.t. they generate $f(x)$.

Def ordered basis : a basis with an order.

Ex: F^3, $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Ex : $P_n(F)$, $\{1, x, x^2, \cdots, x^n\}$

Def: Given an ordered basis $\beta = \{u_1, \cdots, u_n\}$ of V

$\forall x \in V$, $x = a_1 u_1 + \cdots + a_n u_n$

$[x]_\beta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ is the coordinate of x under basis β.

Ex: $P_2(\mathbb{R})$, $\beta = \{1, x, x^2\}$

$f(x) = 4 + 6x - 7x^2$

$[f]_\beta = \begin{bmatrix} 4 \\ 6 \\ -7 \end{bmatrix}$

Def $T: V \longrightarrow W$

$\beta = \{v_1, \cdots, v_n\}$ is an ordered basis of V

$\gamma = \{w_1, \cdots, w_m\}$ - - - - - - W

Then $T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$, $\forall j = 1, \cdots, n$.

$A = [a_{ij}]_{m \times n}$ is the matrix representation of T under bases β and γ.

We write $A = [T]_\beta^\gamma$

$$A = \left[\quad \boxed{\begin{matrix} a_{1j} \\ \vdots \\ a_{mj} \end{matrix}} \quad \right]_{m \times n}$$

$\rotatebox{90}{=}$ $[T(v_j)]_\gamma$ $\quad T(v_j) \in W$.

Ex: $T: P_3(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$ $\quad \beta = \{1, x, x^2, x^3\}$

$f(x) \longmapsto f'(x)$ $\quad \gamma = \{1, x, x^2\}$

$T(1) = 0 = 0\cdot 1 + 0\cdot x + 0\cdot x^2$

$T(x) = 1 = 1\cdot 1 + 0\cdot x + 0\cdot x^2$

$T(x^2) = 2x = 0\cdot 1 + 2\cdot x + 0\cdot x^2$

$T(x^3) = 3x^2 = 0\cdot 1 + 0\cdot x + 3x^2$

$$[T]_\beta^\gamma = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}_{3\times 4}$$

Ex: $I_V : V \longrightarrow V$ $\quad \beta = \{v_1, \dots, v_n\}$

$v \longmapsto v$

$$[I_V]_\beta^\beta = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

Def $\quad T : V \longrightarrow V \quad \beta = \{v_1, \dots, v_n\}$

$[T]_\beta \triangleq [T]_\beta^\beta \qquad [I_V]_\beta = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

is defined as

Def For linear $T_1, T_2 : V \longrightarrow W$

Define $T_1 + T_2 : V \longrightarrow W$ as

① $(T_1 + T_2)(x) \triangleq T_1(x) + T_2(x)$

② $(aT_1)(x) \triangleq a(T_1(x))$, $a \in F$.

Theorem 2.7 V, W are V.S. over same F

$T_1, T_2 : V \longrightarrow W$ are linear.

① $aT_1 + bT_2$ is linear transformation $\forall a, b \in F$

② All linear transformations from V to W forms a vector space over F.

$\mathcal{L}(V,W)$

Theorem 2.8 $T_1, T_2 : \underset{\beta}{V} \longrightarrow \underset{\gamma}{W}$

① $[T_1+T_2]_\beta^\gamma = [T_1]_\beta^\gamma + [T_2]_\beta^\gamma$

② $[aT_1]_\beta^\gamma = a[T_1]_\beta^\gamma$, $a \in F$.

$V : \beta = \{v_1, \cdots, v_n\}$

$W : \gamma = \{w_1, \cdots, w_m\}$

$T : V \longrightarrow W$

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i = [w_1, w_2, \cdots, w_m] \underbrace{\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}}_{= [T(v_j)]_\gamma}$$

a_{ij} are scalars

abstract vectors

$$[T(v_1), \cdots, T(v_j), \cdots, T(v_n)] = [w_1, w_2, \cdots, w_m] \underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & & a_{mj} & & a_{mn} \end{pmatrix}}_{A = [T]_\beta^\gamma}$$

$\forall v \in V$, $[v]_\beta = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

$$\begin{aligned} T(v) &= T(a_1 v_1 + \cdots + a_n v_n) \\ &= a_1 T(v_1) + \cdots + a_n T(v_n) \\ &= \underline{[T(v_1), \cdots, T(v_n)]} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \end{aligned}$$

$$= [w_1 \cdots w_m] \underbrace{\begin{bmatrix} & A & \end{bmatrix}_{m\times n} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}}$$

$[\ ]_{m\times 1}$ is coordinates of $T(v)$ under $\{w_i\}$

$$= [w_1 \cdots w_m] \underbrace{[T]_\beta^\gamma [v]_\beta}_{= [T(v)]_\gamma}$$

$$\Rightarrow \quad [T(v)]_\gamma = [T]_\beta^\gamma [v]_\beta \quad \text{(Theorem 2.14).}$$

Theorem 2.9 V, W, Z are V.S. over F

$T : V \to W$ are linear

$U : W \to Z$

$UT \triangleq U \circ T : V \to Z$ is also linear.

$v \longmapsto U \circ T(v) = U(T(v))$

Theorem 2.10 $T, U_1, U_2 \in \mathcal{L}(V)$ $\mathcal{L}(V) = \mathcal{L}(V, V)$

① $T(U_1 + U_2) = TU_1 + TU_2$

$(U_1 + U_2)T = U_1 T + U_2 T$

② $T(U_1 U_2) = (TU_1)U_2$

③ $TI = IT = T$

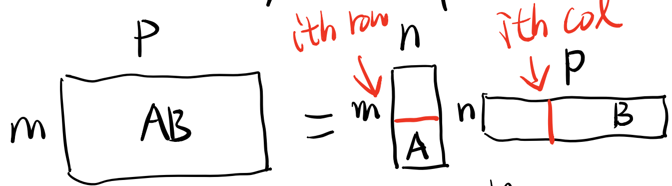
$I_V : V \to V$ is identity map.

④ $\forall a \in F$, $(U_1 U_2) = (aU_1)U_2 = U_1(aU_2)$

Multiplication of matrices

$A \in F^{m\times n}$, $B \in F^{n\times p}$

$AB \in F^{m \times p}$



$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

Theorem 2.11

$T: \underset{\alpha}{V} \longrightarrow \underset{\beta}{W}$ $\quad \alpha, \beta, \gamma$ are ordered bases of V, W, Z

$U: \underset{\beta}{W} \longrightarrow \underset{\gamma}{Z}$

$UT: \underset{\alpha}{V} \longrightarrow \underset{\gamma}{Z}$

$$[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$$

Def $\quad T: V \longrightarrow W$ $\quad$ are linear

$U: W \longrightarrow V$

If $\quad TU = I_W : W \longrightarrow W$

and $UT = I_V : V \longrightarrow V$,

then U is called inverse of T.

① Denote $U = T^{-1}$

② $(TU)^{-1} = U^{-1} T^{-1}$

$(T^{-1})^{-1} = T$

③ $T: V \to W$ $\quad \dim(V) = \dim(W)$ finite

then T is invertible $\Leftrightarrow \operatorname{rank}(T) = \dim(V)$

Theorem 2.5 Assume $\dim(V)=\dim(W)$

$\text{rank}(T)=\dim(V) \iff \begin{cases} \text{1-to-1} \\ \text{onto} \end{cases}$

$\Updownarrow$

Can define an inverse.

Theorem 2.17 If $T: V \to W$ is invertible and linear then $T^{-1}: W \to V$ is linear.

Def V and W are V.S. over F.

We say V is isomorphic to W if there is an invertible linear transformation $T: V \to W$.

an isomorphism from V to W.

Theorem 2.19 Assume V, W are finite dim over F

Then V is isomorphic to $W \iff \dim(V)=\dim(W)$.

Theorem 2.20 V, W are finite dim over F.

$\dim(V)=n$, $\dim(W)=m$

β γ

Define $\Phi_\beta^\gamma : \mathcal{L}(V,W) \longrightarrow F^{m\times n}$

$T \longmapsto [T]_\beta^\gamma$

Then Φ_β^γ is an isomorphism.

Corollary: $\dim(\mathcal{L}(V,W)) = \dim(F^{m\times n})$.

$\dim(\mathcal{L}(\mathbb{R}^2)) = \dim(\mathbb{R}^{2\times 2}) = 4$.

Example:

Geometry	Algebra
$\mathcal{L}(\mathbb{R}^2) = \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$	$\mathbb{R}^{2\times 2}$
$\beta = \gamma = \left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right\}$	
① Rotation by 30° counterclockwise $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{pmatrix}\cos\frac{\pi}{6} & -\sin\frac{\pi}{6}\\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$	$[T]_\beta^\gamma = \begin{pmatrix}\cos\frac{\pi}{6} & -\sin\frac{\pi}{6}\\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6}\end{pmatrix}$
② Reflection w.r.t. x-axis $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$	$[T]_\beta^\gamma = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$
③ Projection to x-axis $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$	$[T]_\beta^\gamma = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$
④ Identity	$[I]_\beta^\gamma = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

$\mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$ is $\underbrace{\text{isomorphic}}_{\text{Theorem 2.20}}$ to $\mathbb{R}^{2\times 3}$

Theorem 2.19 $\Rightarrow$ $\dim(\mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)) = \dim(\mathbb{R}^{2\times 3}) = 6$.

Ex: $\underset{m=3}{V}, \underset{n=2}{W}, \underset{k=2}{Y}, \underset{\ell=3}{Z}$ are V.S. over F

$\mathcal{L}(V, W)$ is isomorphic to $F^{n\times m}$

$\mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$ is isomorphic to $F^{2\times 3}$

$\mathcal{L}(Y, Z)$ - - - - - to $F^{\ell\times k}$

$\mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$ is isomorphic to $F^{3\times 2}$

Can $\mathcal{L}(V,W)$ be isomorphic to $\mathcal{L}(Y,Z)$?

Theorem 2.19 implies

$\mathcal{L}(V,W)$ is isomorphic to $\mathcal{L}(Y,Z)$

if and only if $\dim(\mathcal{L}(V,W)) = \dim(\mathcal{L}(Y,Z))$

$$\Updownarrow$$

$$\dim(F^{n\times m}) = \dim(F^{\ell\times k})$$

$$\Updownarrow$$

$$mn = k\ell.$$

Ex: $\dim\left(\mathcal{L}\left(\underbrace{\mathcal{L}(V,W)}_{\dim = mn}, \underbrace{\mathcal{L}(Y,Z)}_{\dim = k\ell}\right)\right) = mnk\ell.$