Review: the following are equivalent for $A \in F^{n\times n}$:

$L_A: F^n \to F^n$, $v \mapsto Av$

① A^{-1} exists

② $\operatorname{rank}(A) = n$

③ $\operatorname{Nullity}(L_A) = 0$ ($\Leftrightarrow \operatorname{Rank}(L_A) = n$) $\dim(N(L_A)) + \dim(R(L_A)) = n$.

④ $Ax = 0$ has only trivial sol.

⑤ $Ax = b$ has a unique sol $A^{-1}b$.

⑥ row space has dim n.

⑦ col — - - — -.

⑧ $A = E_1 \cdots E_n$ a product of elementary matrices.

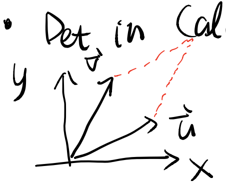
⑨ L_A is an isomorphism

⑩ $\det(A) \neq 0$.

Chapter 4 determinant of square matrices

$\det(A) = |A|$ is a scalar

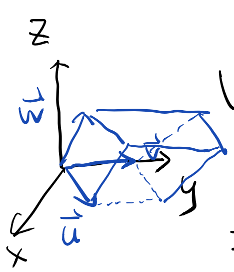
• Det in Calculus



Area of parallelogram generated by $\vec{u} = \begin{pmatrix} a \\ b \end{pmatrix}$, $\vec{v} = \begin{pmatrix} c \\ d \end{pmatrix}$

is $\left| \det\begin{pmatrix} a & c \\ b & d \end{pmatrix} \right| = |ad - bc|$

or $\left| \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right| = |ad - bc|$



Volume of parallelepiped generated by $\vec{u}, \vec{v}, \vec{w}$
$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} d \\ e \\ f \end{pmatrix} \begin{pmatrix} g \\ h \\ i \end{pmatrix}$

$= |\vec{u} \times \vec{v} \cdot \vec{w}|$

$= \left| \det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \right|$

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = +aei + bfg + dhc - ceg - fha - bdi$$

- Geometric Interpretation of $\det(A)$ for L_A.

$$L_A : \mathbb{R}^2 \to \mathbb{R}^2 \qquad \beta = \left\{ \underset{e_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \underset{e_2}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \right\}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto A\begin{pmatrix} x \\ y \end{pmatrix}$$

- $\det(A)$ has a sign : $+$ or $-$
- $|\det(A)|$ is the area of $\square$ generated by two vectors $L_A(e_1)$ and $L_A(e_2)$

$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \qquad |\det(A)| = |ad - bc|$$

$$L_A(e_1) = Ae_1 = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$L_A(e_2) = Ae_2 = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$$

$$T : \mathbb{R}^2 \to \mathbb{R}^2 \qquad \beta = \left\{ \underset{e_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \underset{e_2}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \right\}$$

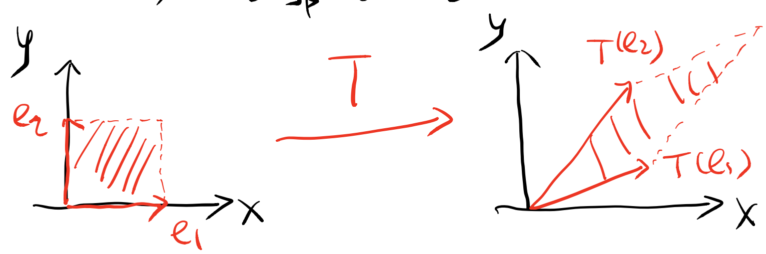
$$v \mapsto T(v)$$

$$v = [v]_\beta$$

$$T(v) = [T(v)]_\beta = [T]_\beta [v]_\beta = [T]_\beta v$$

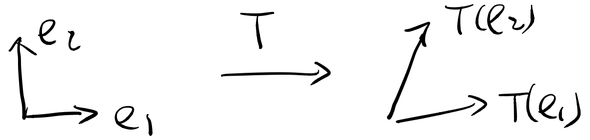
$$T(e_1) = [T]_\beta e_1 = \text{first col of } [T]_\beta$$

$$T(e_2) = [T]_\beta e_2 = \text{second} \text{ - - - -}$$

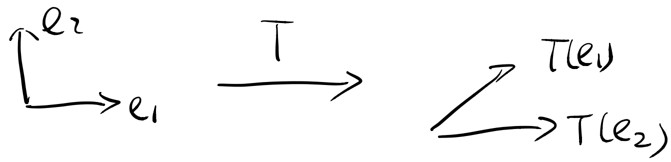


Area $= |\det([T]_\beta)|$

- $|\det([T]_\beta)|$ quantifies how much T changes area.
- The sign of $\det([T]_\beta)$ represents the orientation.
  + or −

① 

positive orientation

② 

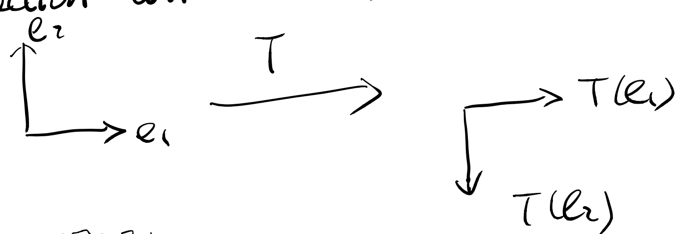
negative orientation.

Ex ① Rotation by θ counterclockwise

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\det\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \cos^2\theta + \sin^2\theta = 1.$$

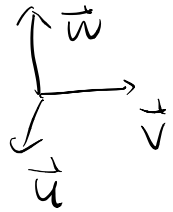
② Reflection w.r.t. x-axis



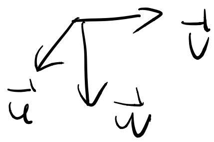
$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\det\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -1$$

- 3D orientation of $\vec{u}, \vec{v}, \vec{w}$
  Right Hand Rule



positive orientation



negative orientation

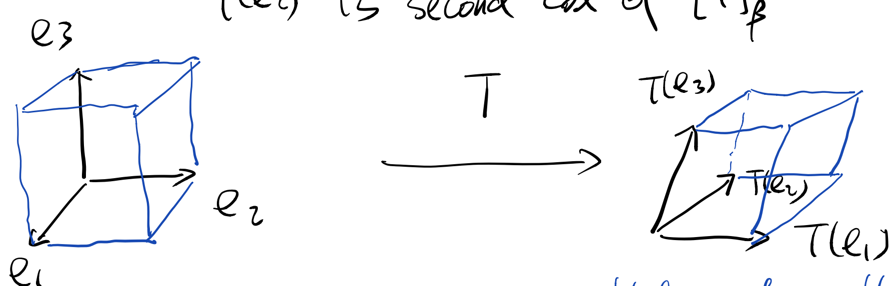
$$\beta = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$v \longmapsto T(v)$$
$$v = [v]_\beta$$

$$T(v) = [T(v)]_\beta = [T]_\beta [v]_\beta = [T]_\beta v$$

$$T(e_1) = [T]_\beta e_1 = \begin{bmatrix} \ \vert & \vert & \vert \ \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \text{first col of } [T]_\beta$$

$T(e_2)$ is second col of $[T]_\beta$



Volume of parallelepiped is

$$|T(e_1) \times T(e_2) \cdot T(e_3)|$$
$$= |\det([T]_\beta)|$$

- $|\det([T]_\beta)|$ quantified how much T changes vol.
- sign $\det[T]_\beta$ means the orientation of $T(e_1), T(e_2), T(e_3)$.

We want to and will define $\det(A)$ for $A \in F^{n \times n}$

Its geometric meaning is the following:

$$T = L_A : F^n \to F^n$$
$$x \mapsto Ax$$

$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\cdots$, $e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

determine an n-dim unit cube.

Volume of image of this unit cube is $|\det(A)|$

and sign of $\det(A)$ is orientation (order)

<u>Def</u> For $A \in F^{n\times n}$, A_{ij} denotes its entry in $\begin{cases} \text{row } i \\ \text{col } j \end{cases}$.

The cofactor matrix of A_{ij} is the $(n-1)\times(n-1)$ matrix obtained by deleting i-th row and j-th col in A.

Ex: $A = \begin{pmatrix} 1 & -1 & 2 & 3 \\ 3 & 4 & \boxed{1} & 2 \\ 2 & -5 & -3 & 8 \\ -6 & 2 & -4 & 1 \end{pmatrix}$ $\qquad A_{23} = 1$

The cofactor matrix of A_{23} is $\begin{pmatrix} 1 & -1 & 3 \\ 2 & -5 & 8 \\ -6 & 2 & 1 \end{pmatrix}$

<u>Def</u> For $A \in F^{n\times n}$, det can be defined recursively

as $\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det(\underbrace{\tilde{A}_{ij}}_{\text{cofactor matrix of } A_{ij}})$ Cofactor expansion along i-th row or $\det(A) = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} \det(\widetilde{A_{ij}})$ cofactor expansion along j-th col

$$\det\begin{pmatrix} 1 & -1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & -5 & -3 & 8 \\ -6 & 2 & -4 & 1 \end{pmatrix} \overset{\text{(2nd row)}}{=} (-1)^{2+1} 3 \cdot \begin{vmatrix} -1 & 2 & 3 \\ -5 & -3 & 8 \\ 2 & -4 & 1 \end{vmatrix}$$

$$+ (-1)^{2+2} 4 \cdot \begin{vmatrix} 1 & 2 & 3 \\ 2 & -3 & 8 \\ -6 & -4 & 1 \end{vmatrix}$$

$$+ (-1)^{2+3} \cdot 1 \cdot \begin{vmatrix} 1 & -1 & 3 \\ 2 & -5 & 8 \\ -6 & 2 & 1 \end{vmatrix}$$

$$+ (-1)^{2+4} \cdot 2 \cdot \begin{vmatrix} 1 & -1 & 2 \\ 2 & -5 & -3 \\ -6 & 2 & -4 \end{vmatrix}$$

Remark: ① For $A \in F^{4\times4}$, compute it by four 3×3 dets.

② For $A \in F^{5\times5}$, compute it by five 4×4 dets.

Facts: ① Type 1 row/col ops changes det by (-1).

② Type 2 row/col ops: multiply a row/col by k will multiply k to det.

③ Type 3 row/col ops: no changes to det.

④ $\det(I) = 1$.

Example: $$\begin{vmatrix} 1 & 3 & -3 \\ -3 & -5 & 2 \\ -4 & 4 & -6 \end{vmatrix} \overset{(3\cdot r_1 + r_2 \to r_2)}{=} \begin{vmatrix} 1 & 3 & -3 \\ 0 & 4 & -7 \\ -4 & 4 & -6 \end{vmatrix}$$

$$(4r_1 + r_3 \to r_3) = \begin{vmatrix} 1 & 3 & -3 \\ 0 & 4 & -7 \\ 0 & 16 & -18 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 1 & 3 & -3 \\ 0 & 4 & -7 \\ 0 & 8 & -9 \end{vmatrix}$$

$$= 2\left[(-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} 4 & -7 \\ 8 & -9 \end{vmatrix} + (-1)^{1+2} \cdot 0 \cdot \left| \quad \right| + (-1)^{1+3} \cdot 0 \left| \quad \right| \right]$$

$$= 2 \begin{vmatrix} 4 & -7 \\ 8 & -9 \end{vmatrix}$$

Example: $\begin{vmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ -2 & -3 & -5 & 2 \\ 4 & -4 & 4 & -6 \end{vmatrix}$

$$\left(C1 \cdot \left(-\tfrac{1}{2}\right) + C4 \to C4\right)$$

$$= \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 3 & -3 \\ -2 & -3 & -5 & 3 \\ 4 & -4 & 4 & -8 \end{vmatrix}$$

$$= 4 \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 3 & -3 \\ -2 & -3 & -5 & 3 \\ 1 & -1 & 1 & -2 \end{vmatrix}$$

$$= 8 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -3 \\ -2 & -3 & -5 & 3 \\ 1 & -1 & 1 & -2 \end{vmatrix}$$

(cofactor expansion along first row)

$$= 8 \cdot (-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} 1 & 3 & -3 \\ -3 & -5 & 3 \\ -1 & 1 & -2 \end{vmatrix}$$

$$= 8 \begin{vmatrix} 1 & 3 & -3 \\ -3 & -5 & 3 \\ -1 & 1 & -2 \end{vmatrix}$$

$$= 8 \begin{vmatrix} 1 & 3 & -3 \\ 0 & 4 & -6 \\ 0 & 4 & -5 \end{vmatrix}$$

$$= 8 \cdot (-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} 4 & -6 \\ 4 & -5 \end{vmatrix}$$

$$= 8 \cdot (-20 + 24) = 32.$$

It is easy to verify that

① If E is type 1 elementary matrix, $\det(E) = -1$.

② - - — - - 2 - - - - - — generated by multiply a col/row by a number k, $\det(E) = k$.

③ - — — 3 - - - — , $\det(E) = 1$.