MA 353 Section 001/015
Spring 2021

Homework 5

Due on Mar 4th before 1pm on gradescope.

1. (20 pts) Compute the determinant of the following matrices

   (a)
$$\begin{pmatrix} 1 & -2 & 3 & -12 \\ -5 & 12 & -14 & 19 \\ -9 & 22 & -20 & 31 \\ -4 & 9 & -14 & 15 \end{pmatrix}$$

   (b)
$$\begin{pmatrix} i & 2 & -1 \\ 3 & 1+i & 2 \\ -2i & 1 & 4-i \end{pmatrix},$$
   where $i = \sqrt{-1}$.

2. (40 pts) Use det to prove the following for $A \in F^{n\times n}$

   (a) If A^k is zero matrix for some positive integer k (such matrices are called nilpotent), then A is not invertible.

   (b) If $A^T = -A$ (such matrices are called skew-symmetric) and n is odd, then A is not invertible.

   (c) If $A^T A = I$ (such matrices are called orthogonal), then $\det(A) = \pm 1$.

   (d) If A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$.

3. (10 pts) Let $\beta = \{v_1, \cdots, v_n\}$ be an ordered basis of a vector space V over the filed F. Let $S = \{u_1, \cdots, u_n\}$ and let A be the matrix consisting of $[u_i]_\beta$ as columns. Prove that S is linearly independent if and only if $\det(A) \neq 0$.

4. (10 pts) Compute $\det(A + tI)$ (here t is a scalar variable) for $A \in F^{n\times n}$ and the identity matrix I of size $n \times n$:
$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & a_0 \\ -1 & 0 & 0 & \cdots & 0 & a_1 \\ 0 & -1 & 0 & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & a_{n-1} \end{pmatrix}$$

5. (10 pts) Find all eigenvalues and eigenvectors for the matrix

$$\begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}.$$

6. (10 pts) Let $A, B, I \in F^{n\times n}$ where I is the identity matrix. Consider a $2n \times 2n$ matrix M:

$$M = \begin{pmatrix} A & B \\ O & I \end{pmatrix},$$

where O denotes zero matrix of size $n \times n$. Prove that $\det(M) = \det(A)$. Hint: discuss it for two cases: 1) $\text{rank}(A) < n$; 2) $\text{rank}(A) = n$. For the second case, it is useful to multiply the matrix $P = \begin{pmatrix} A^{-1} & O \\ O & I \end{pmatrix}$ to M.