MA 353 Section 001/015
Spring 2021

Homework 7

Due on **Mar 23rd Tuesday before 1pm** on gradescope.

1. (10 pts) In $V = C([0,1])$, let $f(t) = t$ and $g(t) = e^t$. Compute

$$\langle f, g\rangle := \int_0^1 f(t)g(t)dt,$$

$\|f\|$, $\|g\|$ and $\|f+g\|$. Then verify (you can use calculator if needed) both the Cauchy-Schwart inequality and the triangle inequality for f, g.

2. (20 pts) The Cauchy-Schartz inequality $|\langle x, y\rangle| \leq \|x\|\|y\|$ and the triangle inequality $\|x+y\| \leq \|x\| + \|y\|$have many different explicit forms, depending on what the abstract vectors x, y are and how the inner product is defined. For example, consider the following two inequalities that we are familiar with:

$$\left|\sum_{i=1}^{n} a_i\bar{b}_i\right| \leq \left[\sum_{i=1}^{n}|a_i|^2\right]^{\frac{1}{2}}\left[\sum_{i=1}^{n}|b_i|^2\right]^{\frac{1}{2}}, \quad \forall a_i, b_i \in \mathbb{C},$$

$$\left[\sum_{i=1}^{n}|a_i+b_i|^2\right]^{\frac{1}{2}} \leq \left[\sum_{i=1}^{n}|a_i|^2\right]^{\frac{1}{2}} + \left[\sum_{i=1}^{n}|b_i|^2\right]^{\frac{1}{2}}, \quad \forall a_i, b_i \in \mathbb{C}.$$

These two inequalities above are exactly the Cauchy-Schartz inequality and the triangle inequality in the vector space $V = \mathbb{C}^n$ with Standard inner product. Now prove the following inequalities by showing that they are (or implied by) the Cauchy-Schartz inequality and the triangle inequality for some vector space with some inner product (specify what the vector space and the inner product are):

(a)

$$\left|\int_0^1 f(x)\bar{g}(x)dx\right| \leq \left[\int_0^1 |f(x)|^2dx\right]^{\frac{1}{2}}\left[\int_0^1 |g(x)|^2dx\right]^{\frac{1}{2}},$$

$$\left[\int_0^1 |f(x)+g(x)|^2dx\right]^{\frac{1}{2}} \leq \left[\int_0^1 |f(x)|^2dx\right]^{\frac{1}{2}} + \left[\int_0^1 |g(x)|^2dx\right]^{\frac{1}{2}},$$

where $f(x), g(x)$ are two continuous complex-valued functions defined on the interval $x \in [0,1]$.

(b)

$$tr(AB) \le \sqrt{tr(A^2)}\sqrt{tr(B^2)},$$
$$\sqrt{tr((A+B)^2)} \le \sqrt{tr(A^2)} + \sqrt{tr(B^2)},$$

where A and B are two real symmetric $n \times n$ matrices.

3. (20 pts) Let V be an inner product space over F. Prove the following:
   (a) *Parallelogram law*: if $F = \mathbb{C}$,
   $$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2, \forall x, y \in V.$$
   (b) *Polar identity*: if $F = \mathbb{R}$, $\langle x, y\rangle = \frac{1}{4}\|x+y\|^2 - \frac{1}{4}\|x-y\|^2, \forall x, y \in V$.
   (c) *Polar identity*: if $F = \mathbb{C}$, $\langle x, y\rangle = \frac{1}{4}\sum_{k=1}^{4} i^k \|x + i^k y\|^2, \forall x, y \in V$. Here $i = \sqrt{-1}$.
   (d) $|\|x\| - \|y\|| \le \|x - y\|, \forall x, y \in V$.
4. (20 pts) For $V = F^n$ with standard inner product and $A \in F^{n\times n}$ (where $F = \mathbb{C}$ or $\mathbb{R}$):
   (a) Prove that $\langle x, Ay\rangle = \langle A^*x, y\rangle, \forall x, y \in V$.
   (b) Assume $\langle x, Ay\rangle = \langle Bx, y\rangle, \forall x, y \in V$ for some $B \in F^{n\times n}$. Prove that $B = A^*$.
   (c) For any orthonormal basis β for V, let Q be the matrix whose columns are vectors in β. Prove that $Q^* = Q^{-1}$.
   (d) Define two linear operators $T : V \longrightarrow V$ and $U : V \longrightarrow V$ by $T(x) = Ax$ and $U(x) = A^*x$. Prove that $[U]_\beta = [T]_\beta^*$ for any orthonormal basis β for V.
5. (20 pts) Apply Gram-Schmidt process to the given set S of the inner product space V to obtain an orthogonal basis for $span(S)$. Then normalize the vectors in this basis to obtain an orthonormal basis β for $span(S)$.
   (a) $V = \mathbb{R}^4$, standard inner product, $S = \left\{ \begin{pmatrix} 1 \\ -2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 2 \\ 8 \end{pmatrix} \right\}$.
   (b) $V = C([0,\pi])$ over $F = \mathbb{R}$, with $\langle f, g\rangle = \int_0^\pi f(t)g(t)dt$. $S = \{\sin t, \cos t, 1, t\}$. Feel free to use computer or online tools for computing integrals.
6. (10 pts) For $A \in \mathbb{F}^{m\times n}$ ($F = \mathbb{C}$ or $\mathbb{R}$), prove that $(R(L_{A^*}))^\perp = N(L_A)$.