MA 353 Section 001/015
Spring 2021

Homework 11

Due on **April 29th Thursday before 1pm** on gradescope.

Recall that we only consider $F = \mathbb{R}$ or $\mathbb{C}$ whenever inner product is involved.

1. (10 pts) For a finite dimensional inner product space V over $F = \mathbb{C}$, let β and γ be two orthonormal bases, consider the matrix $Q = [I]_\beta^\gamma$. Prove that L_Q is unitary. (thus we can further prove Q is unitary if we want.)
Hint: Let $\beta = \{v_1, \cdots, v_n\}$ be an orthonormal basis. $\forall x \in V$, assume $x = \sum_{i=1}^{n} a_i v_i$, recall that we have proven $\|x\|^2 = \sum_{i=1}^{n} |a_i|^2$.

2. (20 pts) Let A be a real $n \times n$ matrix. Prove that A is symmetric if and only if A is orthogonally equivalent to a real diagonal matrix.
Hint: this is the real analog of Theorem 6.19, you can prove it by following the proof of Theorem 6.19.

3. (30 pts) Consider the matrix

$$A = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

   (a) (10 pts) Find all eigenvalues, their algebraic multiplicity and geometrical multiplicity, and basis vectors for all eigenspaces.

   (b) (10 pts) For this particular matrix, there is one eigenvalue λ_2 for which geometrical multiplicity is less than algebraic multiplicity. This ensures existence of one generalized eigenvector: let v be its eigenvector, then find the generalized eigenvector u defined as solution to the nonhomogeneous linear system $(A - \lambda_2 I)u = v$.

   (c) (10 pts) For this particular matrix, there are two distinct eigenvalues λ_1 and λ_2. Let v_1 be eigenvector for λ_1. Form a matrix $Q = [v_1 \ v \ u]$. Then by the definition of eigenvectors and generalized eigenvectors, we have

$$AQ = [Av_1 \ Av \ Au] = [\lambda_1 v_1 \ \lambda_2 v \ \lambda_2 u + v] = [v_1 \ v \ u] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix}.$$

Here $J = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix}$ is called Jordan Form of A. Find the explicit expression of J, Q, Q^{-1} and verify that $A = QJQ^{-1}$ (and this is what eigenvalue decomposition looks like for a nondiagonalizable matrix).

4. (10 pts) Background: *Any real matrix* $A \in \mathbb{R}^{n\times n}$ *has a Singular Value Decomposition (SVD) in the form that* $A = U\Sigma V^T$ *where* U, V *are orthogonal matrices (their columns are called singular vectors) and* Σ *is a diagonal matrix with real non-negative diagonal entries* σ_i *(singular values).* Now let us assume $A \in R^{n\times n}$ is given as $A = U\Sigma V^T$ where $U, V \in R^{n\times n}$ are orthogonal matrices and $\Sigma \in R^{n\times n}$ is a diagonal matrix with diagonal entries $\sigma_i \geq 0$ $(i = 1, \cdots, n)$. Prove that

   (a) σ_i^2 are eigenvalues of AA^T with eigenvectors u_i, columns of U.

   (b) σ_i^2 are eigenvalues of A^TA with eigenvectors v_i, columns of V.

5. (20 pts) Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

   Find its SVD $A = U\Sigma V^T$ by computing σ_i^2 as eigenvalues of AA^T (or A^TA), computing columns u_i of U as orthonormal eigenvectors of AA^T and columns v_i of V as orthonormal eigenvectors of A^TA. **And order them so that** $Av_i = \sigma_i u_i$**.** Finally verify that $A = U\Sigma V^T$.

6. (10 pts) Background: *Any complex matrix* $A \in \mathbb{C}^{n\times n}$ *has a Singular Value Decomposition (SVD) in the form that* $A = U\Sigma V^*$ *where* U, V *are unitary matrices (their columns are called singular vectors) and* Σ *is a diagonal matrix with* **real** *non-negative diagonal entries* σ_i *(singular values).* σ_i *are square root of eigenvalues of* AA^* *(or* A^*A*). Columns of* U *are orthonormal eigenvectors of* AA^**. Columns of* V *are orthonormal eigenvectors of* A^*A*.*
   Let $A \in \mathbb{C}^{n\times n}$. Let σ_i be its singular values and λ_i be its eigenvalues. Prove that we have $\sigma_i = |\lambda_i|$ for a normal matrix A.
   Hint: Plug in the eigenvalue decomposition $A = QDQ^*$ into AA^* and A^*A.