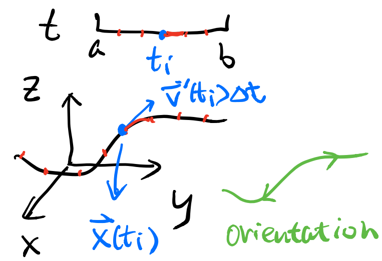
Curve : $\vec{C}(t) = \underbrace{\langle x(t), y(t), z(t) \rangle}_{\vec{x}(t)}$, $t \in [a,b]$

$\vec{V}'(t) = \langle x'(t), y'(t), z'(t) \rangle$

$V(t) = \|\vec{V}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}$



Length of Curve $\boxed{\lim_{\Delta t \to 0} \sum_i V(t_i) \Delta t}$

$$\| \qquad \int_{\vec{C}} ds = \int_a^b V(t)\,dt$$

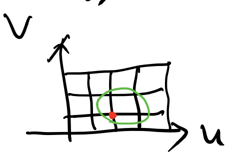
$d\vec{s} = \langle x'(t), y'(t), z'(t) \rangle dt$

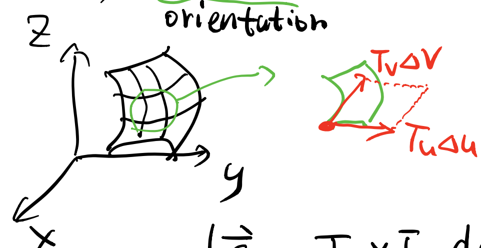
$= \langle dx, dy, dz \rangle$

$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt$

Surface $S \begin{cases} x = f(u,v) \\ y = g(u,v) \\ z = h(u,v) \end{cases}$, $(u,v) \in D$, $\underline{T_u \times T_v}$ orientation

Area of Surface





$\boxed{\lim_{\substack{\Delta u \to 0 \\ \Delta v \to 0}} \sum \|T_u \times T_v\| \Delta u \Delta v}$

$$\iint_S dS$$

$$= \iint_D \|T_u \times T_v\|\, du\,dv$$

$d\vec{S} = T_u \times T_v\, du\,dv$

$dS = \|T_u \times T_v\|\, du\,dv$

Two Types of Line/Surface Integral

$f(x,y,z)$, $\vec{F} = \langle F, G, H \rangle$

Curve

Surface

Type I (orientation independent)

$\int_{\vec{C}} f\, ds = \int_a^b f(x(t), y(t), z(t))\, V(t)\, dt$

$\iint_S f\, dS$

$= \iint_D f(x(u,v), y(u,v), z(u,v)) \underline{\|T_u \times T_v\|}\, du\,dv$

$\int_{-\vec{C}} f ds = \int_{\vec{C}} f ds$

$\iint_{-S} f\, dS = \iint_S f\, dS$

Type II (orientation dependent)

$\int_{\vec{C}} \vec{F} \cdot d\vec{s} = \int_{\vec{C}} \langle F, G, H \rangle \cdot \langle dx, dy, dz \rangle$

$= \int_{\vec{C}} F dx + G dy + H dz$

$\iint_S \vec{F} \cdot d\vec{S} = \iint$

$$= \int_a^b [F x'(t) + G y'(t) + H z'(t)]\, dt$$

$$\int_{-\vec{C}} \vec{F} \cdot d\vec{s} = -\int_{\vec{C}} \vec{F} \cdot d\vec{s}$$

$$d\vec{S} = \vec{n}\, dS$$

$$\vec{n} = \frac{T_u \times T_v}{\| T_u \times T_v \|}$$

$$d\vec{S} = T_u \times T_v\, du\, dv$$

$$= \frac{T_u \times T_v}{\| T_u \times T_v \|} \underbrace{\| T_u \times T_v \|\, du\, dv}_{dS}$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S F\, dy \wedge dz + G\, dz \wedge dx + H\, dx \wedge dy$$

$$\shortparallel$$

$$\iint_S \boxed{\vec{F} \cdot \vec{n}}\, dS$$

$$\iint_{-\vec{S}} \vec{F} \cdot d\vec{S} = -\iint_{\vec{S}} \vec{F} \cdot d\vec{S}$$

$$\shortparallel$$

$$\iint_D \vec{F} \cdot (T_u \times T_v)\, du\, dv$$

$$\shortparallel$$

$$\iint_D \vec{F} \cdot \vec{n}\, \| T_u \times T_v \|\, du\, dv$$

Type I $\iint_S f\, dS$

Type II $\iint_S \vec{F} \cdot \vec{n}\, dS$

Ex: Surface Area of unit sphere

Sol: $S \begin{cases} x = \sin\phi \cos\theta \\ y = \sin\phi \sin\theta \\ z = \cos\phi \end{cases}$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$

$$T_\phi = \langle \cos\phi \cos\theta, \cos\phi \sin\theta, -\sin\phi \rangle$$

$$T_\theta = \langle -\sin\phi \sin\theta, +\sin\phi \cos\theta, 0 \rangle$$

$$T_\phi \times T_\theta = \begin{vmatrix} i & j & k \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & +\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= \langle \sin^2\phi\cos\theta,\ \sin^2\phi\sin\theta,\ \cos\phi\sin\phi \rangle$$

$$\|T_\phi \times T_\theta\| = \sqrt{\sin^4\phi(\cos^2\theta + \sin^2\theta) + \cos^2\phi\sin^2\phi}$$

$$= \sqrt{\sin^2\phi\,(\sin^2\phi + \cos^2\phi)}$$

$$= |\sin\phi| = \sin\phi$$

$$\iint_S dS = \int_0^{2\pi}\left[\int_0^{\pi} \|T_\phi \times T_\theta\|\, d\phi\right] d\theta = \int_0^{2\pi}\left[\int_0^{\pi} \sin\phi\, d\phi\right] d\theta$$

$$= 2\pi \int_0^{\pi} \sin\phi\, d\phi = 4\pi$$