$$\iiint_V f(x,y,z)\,\underbrace{dx\wedge dy\wedge dz}_{\text{orientation}} \underset{\text{(is defined as)}}{:=} \iiint_V f(x,y,z)\,dx\,dy\,dz$$

$V$ is a 3D solid region (Ex: unit ball)

Physical meaning: if $f(x,y,z)$ is density, then $\iiint_V f\,dx\,dy\,dz$ is the total mass.

Ex: $\iiint_V 1\cdot dx\,dy\,dz$ is the volume of $V$.
(limit of Riemann Sum)

Ex: $\iiint_V f\,\underline{dy\wedge dx\wedge dz} = -\iiint_V f\,dx\wedge dy\wedge dz = -\iiint_V f\,dx\,dy\,dz$

Ex: Compute volume of ball with radius $R$.

Sol: Region $V$: $x^2+y^2+z^2\le R$ ($\rho\le R$, $0\le\phi\le\pi$, $0\le\theta\le 2\pi$)

$$\iiint_V 1\,dx\,dy\,dz = \iiint_{x^2+y^2+z^2\le R} 1\cdot dx\,dy\,dz = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \underline{\rho^2\sin\phi}\,d\rho\,d\phi\,d\theta$$

$$= 2\pi\left[\int_0^{\pi}\sin\phi\,d\phi\right]\left[\int_0^R \rho^2\,d\rho\right]$$

$$= 2\pi\cdot\left[-\cos\phi\,\Big|_0^{\pi}\right]\cdot\left[\tfrac{1}{3}\rho^3\,\Big|_0^R\right] = \tfrac{4}{3}\pi R^3$$

---

Jacobian
$$\begin{cases} dx\,dy = \underline{r\,dr\,d\theta} \\ dx\,dy\,dz = \underline{\rho^2\sin\phi}\,d\rho\,d\phi\,d\theta \end{cases}$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{aligned} dx &= \cos\theta\,dr - r\sin\theta\,d\theta \\ dy &= \sin\theta\,dr + r\cos\theta\,d\theta \end{aligned}$$

$$dx\wedge dy = r\cos^2\theta\,dr\wedge d\theta - r\sin^2\theta\,d\theta\wedge dr = r(\cos^2\theta+\sin^2\theta)\,dr\wedge d\theta$$
$$= r\,dr\wedge d\theta$$

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases} \qquad \begin{aligned} dx &= \boxed{\sin\phi\cos\theta\,d\rho} + \rho\cos\phi\cos\theta\,d\phi - \rho\sin\phi\sin\theta\,d\theta \\ dy &= \sin\phi\sin\theta\,d\rho + \rho\cos\phi\sin\theta\,d\phi + \rho\sin\phi\cos\theta\,d\theta \\ dz &= \cos\phi\,d\rho - \rho\sin\phi\,d\phi \end{aligned}$$

$$dx\wedge dy\wedge dz = -\rho^2\sin^3\phi\cos^2\theta\,d\rho\wedge d\theta\wedge d\phi + \rho^2\cos^2\phi\sin\phi\cos^2\theta\,d\phi\wedge d\theta\wedge d\rho$$

$$+\rho^2\sin^3\phi\sin^2\theta\, d\theta\wedge d\rho\wedge d\phi - \rho^2\cos^2\phi\sin\phi\sin^2\theta\, d\theta\wedge d\phi\wedge d\rho$$

$$= \rho^2\sin^3\phi\cos^2\theta\, d\rho\wedge d\phi\wedge d\theta + \rho^2\cos^2\phi\sin\phi\cos^2\theta\, d\rho\wedge d\phi\wedge d\theta$$
$$+\rho^2\sin^3\phi\sin^2\theta\, d\rho\wedge d\phi\wedge d\theta + \rho^2\cos^2\phi\sin\phi\sin^2\theta\, d\rho\wedge d\phi\wedge d\theta$$

$$= \rho^2\sin^3\phi\, d\rho\wedge d\phi\wedge d\theta + \rho^2\cos^2\phi\sin\phi\, d\rho\wedge d\phi\wedge d\theta$$

$$= \rho^2\sin\phi\left[\sin^2\phi + \cos^2\phi\right] d\rho\wedge d\phi\wedge d\theta$$

$$= \rho^2\sin\phi\, d\rho\wedge d\phi\wedge d\theta$$

---

$$\iiint_V 1\, dx\,dy\,dz \left(= \iiint_V \rho^2\sin\phi\, d\rho\wedge d\phi\wedge d\theta\right) = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$V: 0\le\rho\le R,\ 0\le\phi\le\pi,\ 0\le\theta\le 2\pi$

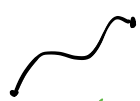
---

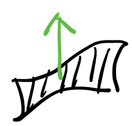
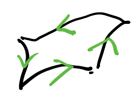
Stokes–Cartan Theorem

$\Omega$ is an $n$-dimensional manifold (a general name for "surface")

$\partial\Omega$ denotes the boundary of $\Omega$ and it is $(n-1)$-dimensional.

Ex: ① $\Omega$ is an interval $[a, b]$, $\partial\Omega$ are two end points.
1D ... 0D ... $b$, $-a$

② $\Omega$ is a curve , $\partial\Omega$ are two end points.

③ $\Omega$ is a surface , $\partial\Omega$ is a curve 
2D ... 1D

④ $\Omega$ is the unit solid ball, $\partial\Omega$ is the unit sphere.
3D ... 2D

⑤ $\Omega$ is defined as $\{(x,y,z,t): x^2+y^2+z^2+t^2\le 1\}$,
4D ball
then $\partial\Omega$ is 3-dimensional.

Suppose $\omega$ is an $\underline{(n-1)}$-form, then $d\omega$ is $\underline{n\text{-form}}$.

Stokes–Cartan Theorem $\displaystyle\int_{\partial\Omega} \omega = \int_{\Omega} d\omega$ with matching orientation for $\partial\Omega$ and $\Omega$.

① $\underline{F(b) - F(a)} = \underline{\int_a^b F'(x)\,dx}$