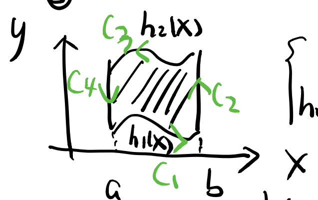
Sketchy Proof for Green's / Gauss / Stokes Theorem (FYI only)

I. Green's Theorem on a plane region D

① D is a rectangle Page 8 of notes.

② D is x-simple : $\iint_D [-P_y]\,dxdy = \int_C P\,dx$




$\begin{cases} a \le x \le b \\ h_1(x) \le y \le h_2(x) \end{cases}$ RHS $= \int_{C_1} P\,dx + \int_{C_3} P\,dx$ (because $dx = 0$ on $C_2, C_4$)

$$\text{LHS} = \int_a^b \left[\int_{h_1(x)}^{h_2(x)} -P_y\,dy\right] dx = \int_a^b -P(x,h_2(x)) + P(x,h_1(x))\,dx$$

$$= \int_a^b P(x,h_1(x))\,dx - \int_a^b P(x,h_2(x))\,dx$$

$C_1 : \begin{cases} x = x \\ y = h_1(x) \end{cases}, a \le x \le b \Rightarrow \int_{C_1} P\,dx = \int_a^b P(x,h_1(x))\,dx$

$-C_3 : \begin{cases} x = x \\ y = h_2(x) \end{cases}, a \le x \le b \Rightarrow \int_{-C_3} P\,dx = \int_a^b P(x,h_2(x))\,dx$

$$\int_C P\,dx = \int_{C_1} P\,dx + \int_{C_3} P\,dx = \int_{C_1} P\,dx - \int_{-C_3} P\,dx$$

③ D is y-simple : $\iint_D Q_x\,dxdy = \int_C Q\,dy$

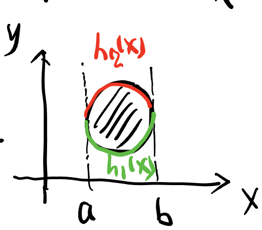

$g_1(y) \le x \le g_2(y)$

$c \le y \le d$

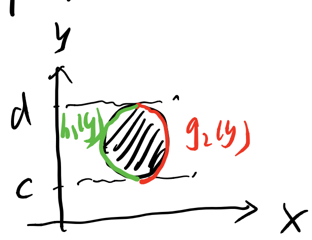
④ D is both x-simple and y-simple : add them up.

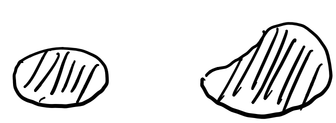
Ex: a disk

X-Simple

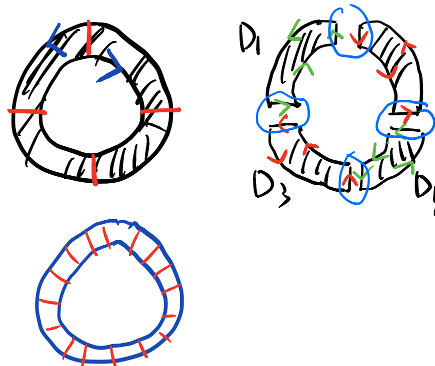



y-Simple



⑤ D is a general region: cut it into pieces




$$\iint_{D_i} [Q_x - P_y]\,dx\,dy = \int_{C_i} P\,dx + Q\,dy$$

$i = 1, 2, 3, 4$

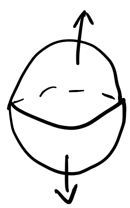
Sum $\Rightarrow$ $\displaystyle\iint_D [Q_x - P_y]\,dx\,dy = \int_C P\,dx + Q\,dy$

## II. Gauss / Divergence Theorem

$$\iiint_V f_x + g_y + h_z \,dx\,dy\,dz = \iint_S \langle f, g, h\rangle \cdot \vec{n}\,dS$$

x-simple, y-simple, z-simple

Consider a ball

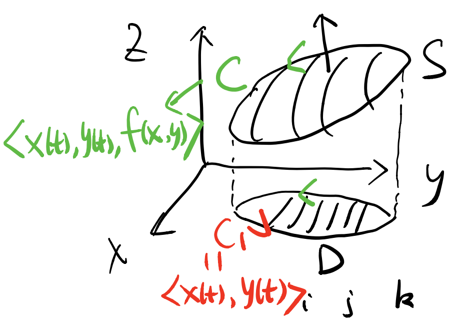



$$\iiint_V h_z\,dz\,dx\,dy$$

$$= \iint_{x^2+y^2\le 1} \left[\int_{-\sqrt{x^2+y^2}}^{\sqrt{x^2+y^2}} h_z\,dz\right] dx\,dy$$

$$= \iint_{x^2+y^2\le 1} \Big[\underbrace{h(x, y, \sqrt{x^2+y^2})}_{z=\sqrt{x^2+y^2}} - \underbrace{h(x, y, -\sqrt{x^2+y^2})}_{z=-\sqrt{x^2+y^2}}\Big]\,dx\,dy$$

## III. Stokes Theorem on a surface $z = f(x, y)$




$S: \begin{cases} x = u \\ y = v \\ z = f(u,v) \end{cases}$, $(u,v) \in D$, upward normal

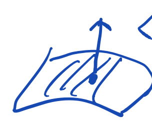
$$\iint_S \nabla \times \langle F, G, H\rangle \cdot \vec{n}\,dS = \oint_C \langle F, G, H\rangle \cdot d\vec{s}$$

$T_u = \langle 1, 0, f_u\rangle$

$T_v = \langle 0, 1, f_v\rangle$

$T_u \times T_v = \underline{\langle -f_u, -f_v, 1\rangle}$

$z = f(x, y)$




Normal vector to tangent plane of a surface

$\{ \; z = f(x,y)$ : $\langle f_x, f_y, -1\rangle$

$F(x,y,z)=C: \langle F_x, F_y, F_z\rangle$

$$LHS = \iint_D \nabla \times \langle F,G,H\rangle \cdot (T_u \times T_v)\, du\, dv$$

$$= \iint_D \underbrace{\langle H_y - G_z,\ F_z - H_x,\ G_x - F_y\rangle \cdot \langle -f_x, -f_y, 1\rangle}\, dx\, dy$$

$$P(x,y) = \langle F, G, H\rangle \cdot \langle 1, 0, f_x\rangle = F(x,y,f(x,y)) + H(x,y,f(x,y)) \cdot f_x(x,y)$$

$F(x,y,f(x,y))$

$$Q(x,y) = \langle F,G,H\rangle \cdot \langle 0,1,f_y\rangle = G + H f_y$$

$$P_y = F_y + F_z \cdot f_y + [H_y + H_z \cdot f_y] \cdot f_x + H \cdot f_{xy}$$

$$Q_x = G_x + G_z f_x + [H_x + H_z \cdot f_x] f_y + H f_{yx}$$

$$Q_x - P_y = \underbrace{(G_x - H_y) - f_x (H_y - G_z) - f_y (F_z - H_x)}$$

$$\int_C \langle F, G, H\rangle \cdot d\vec{s} = \int_C \langle F, G, H\rangle \cdot \langle \underbrace{dx}_{x'(t)dt}, \underbrace{dy}_{y'(t)dt}, dz\rangle$$

$z = f(x(t), y(t))$

$[f_x \cdot x'(t) + f_y \cdot y'(t)]\, dt$

$$= \int_{C_1} P\,dx + Q\,dy$$