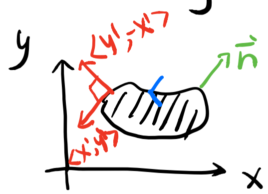
Gauss /Divergence Theorem $\iiint_V \nabla\cdot\langle f,g,h\rangle\, dxdydz = \iint_S \langle f,g,h\rangle\cdot\vec{n}\, dS$

2D Gauss /Divergence Theorem $\iint_D \underbrace{\nabla\cdot\langle f,g\rangle}_{f_x+g_y}\, dxdy = \underline{\int_C \underline{\langle f,g\rangle}\cdot\vec{n}\, ds}$



(unit normal vector to the tangent line outward)

Proof: $\iint_D [Q_x - P_y]\, dxdy = \int_C Pdx + Qdy$

$P = -g\ ,\ Q = f$

LHS in Green's Thm $= \iint_D (f_x + g_y)\, dxdy$

RHS in Green's Thm $= \int_C -gdx + fdy$

$C(t) = \langle x(t), y(t)\rangle\ ,\quad a \le t \le b$

$$= \int_a^b -g\, x'(t)dt + f\, y'(t)dt$$

$$= \int_a^b \left(-g\frac{x'}{\sqrt{}} + f\frac{y'}{\sqrt{}}\right)\cdot \underline{\sqrt{[x'(t)]^2+[y'(t)]^2}\, dt}$$

$$= \int_S \left(-g\frac{x'}{\sqrt{}} + f\frac{y'}{\sqrt{}}\right) ds$$

$$= \int_S \langle f,g\rangle\cdot \underbrace{\left\langle \frac{y'}{\sqrt{}}, \frac{-x'}{\sqrt{}}\right\rangle}_{} ds \rightarrow \text{unit outward normal.}$$

$\langle y', -x'\rangle \perp \langle \underbrace{x', y'}_{\text{tangent vector}}\rangle \Rightarrow \langle y', -x'\rangle$ is a normal vector

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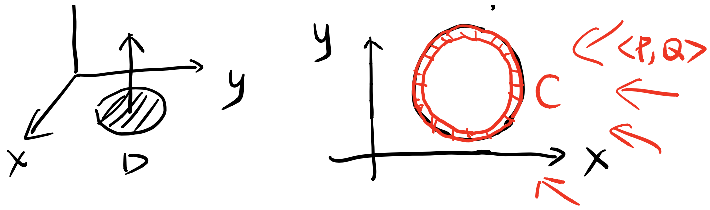
Physical meaning of $\underbrace{\text{Curl}}_{\text{rotation}}$ and $\underbrace{\text{Divergence}}_{\text{compression/expansion}}$

① Curl

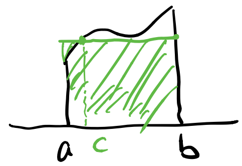
$$\iint_D \underbrace{\nabla\times\langle P,Q,0\rangle\cdot\langle 0,0,1\rangle}_{F(x,y)}\, dxdy = \int_C \underline{\langle P,Q,0\rangle\cdot d\vec{s}}$$

Work done by $\langle P,Q,0\rangle$ along C.

z ↑ D is a flat disk centered at (x_0, y_0)



Mean Value Theorem $\int_a^b f(x)\,dx = f(c)(b-a)$, for some $c \in (a,b)$



Mean Value Theorem $\Rightarrow$ $\iint_D F(x,y)\,dxdy = F(x^*,y^*)\,\text{Area}(D)$

for some point (x^*,y^*) in D

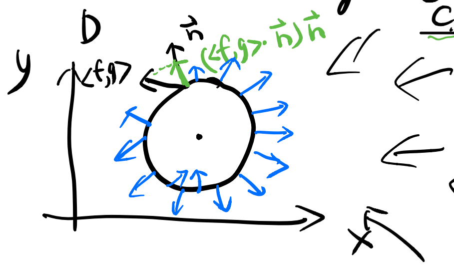
$$\Rightarrow \quad F(x^*,y^*) = \frac{1}{\text{Area}(D)} \int_C \langle P,Q,0\rangle \cdot d\vec{s}$$

Let $D \to (x_0,y_0)$, then $(x^*,y^*) \to (x_0,y_0)$

$$\Rightarrow \quad \nabla \times \langle P,Q,0\rangle \cdot \langle 0,0,1\rangle \Big|_{(x_0,y_0)} = \lim_{\text{Area}(D)\to 0} \frac{1}{\text{Area}(D)} \int_C \langle P,Q,0\rangle \cdot d\vec{s}$$

② Div

$$\iint_D (f_x+g_y)\,dxdy = \int_C \langle f,g\rangle \cdot \vec{n}\,ds$$



The type I line integral of $\langle f,g\rangle \cdot \vec{n}$ quantifies total amount of fluid flowing out of C.

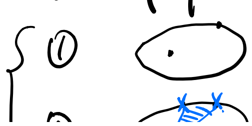
$\langle f,g\rangle$

$$f_x+g_y = \lim_{\text{Area}(D)\to 0} \frac{1}{\text{Area}(D)} \int_C \langle f,g\rangle \cdot \vec{n}\,ds$$

Johannes Kepler 1589–1594

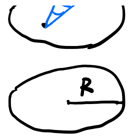
3 Laws of planetary motion

①



Isaac Newton 1642–1726

$F = ma$

② 

③ $\dfrac{T^2}{R^3} = \text{Constant}$

Foundametal Theorem of Calucals

$$\iint_D (Q_x - P_y)\,dx\,dy = \int_C P\,dx + Q\,dy$$

$$\iint_S \nabla\times\langle f,g,h\rangle\cdot\vec{n}\,dS = \int_C \langle f,g,h\rangle\cdot d\vec{s}$$

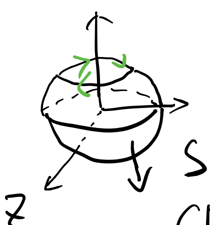
$$\iiint_V \nabla\cdot\langle f,g,h\rangle\,dx\,dy\,dz = \iint_S \langle f,g,h\rangle\cdot\vec{n}\,dS$$

$$\iint_D (f_x + g_y)\,dx\,dy = \int_C \langle f,g\rangle\cdot\vec{n}\,ds$$

$$\int_M d\omega = \int_{\partial M}\omega$$

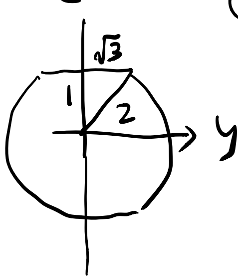
Stokes-Cartan Theorem

HW #10 P4 $\quad \iint_S z\,dy\wedge dz \ \left(\overset{①}{=} \iint_D \langle z,0,0\rangle\cdot T_u\times T_v\,du\,dv\right)$



② $\nabla\cdot\langle z,0,0\rangle = 0 \Rightarrow \begin{cases}\omega = z\,dy\wedge dz\\ d\omega = 0\end{cases} \Rightarrow \omega$ is exact

(Poincare's) $\Rightarrow \omega = d\left(-\frac{z^2}{2}dy\right)$



$$\iint_S \omega = \iint_S d\alpha = \int_C \alpha = -\int_{-C}\alpha = -\int_{-C} -\frac{z^2}{2}dy$$

$$-C : \begin{cases} x = \sqrt{3}\cos t\\ y = \sqrt{3}\sin t\\ z = 1\end{cases}$$

③ 

$$\iint_{S_1+S}\omega = \iiint_V d\omega = 0$$

$$\Rightarrow \iint_S \omega = -\iint_{S_1}\omega$$