Div and Curl

Xiangxiong Zhang

Math Dept, Purdue Univeristy

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Meaning of derivatives

Scalar-valued functions:

1. $f'(x_0)$ is the slope of the tangent line to the curve $y = f(x)$ at $(x_0, f(x_0))$.
2. $z = f(x, y)$ is a surface and $< f_x, f_y, -1 >$ is the normal vector to the tangent plane.
3. $F(x, y, z) = C$ is a level surface and $\nabla F = \langle F_x, F_y, F_z \rangle$ is the normal vector to the tangent plane.
4. $f(x, y) = C$ is a level curve and $\nabla f = \langle f_x, f_y \rangle$ is a vector orthogonal to the tangent vector.

If a vector field $\mathbf{F}(x, y, z) = \langle f(x, y, z), g(x, y, z), h(x, y, z) \rangle$ denotes the velocity of some fluid (water/oil/air):

- $\text{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \langle \partial_x, \partial_y, \partial_z \rangle \cdot \langle f, g, h \rangle = f_x + g_y + h_z$.
  1. Divergence quantifies the compression ($\text{div}\mathbf{F} < 0$) or expansion ($\text{div}\mathbf{F} > 0$).
  2. $\text{div}\mathbf{F} = 0$: incompressible flow such as water/oil at normal temperature.
- $\text{curl}\mathbf{F} = \nabla \times \mathbf{F} = \langle \partial_x, \partial_y, \partial_z \rangle \times \langle f, g, h \rangle = \langle h_y - g_z, f_z - h_x, g_x - f_y \rangle$ quantifies the rotation of the flow.

Why: Green's/Divergence (Gauss)/Stokes Theorems.

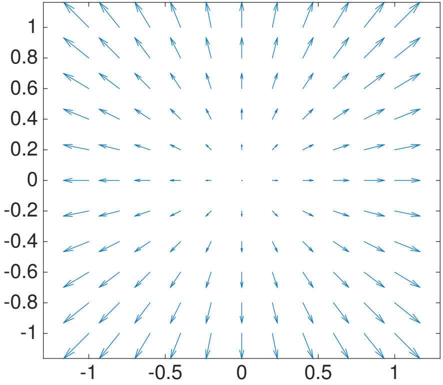
2D Vector fields

Let $\mathbf{U}(x,y) = \langle u(x,y), v(x,y) \rangle$ be a 2D vector field denoting velocity of some 2D flow.

- $\text{div}\mathbf{U} = u_x + v_y$
- $\text{curl}\,\mathbf{U} = \langle \partial_x, \partial_y, \partial_z \rangle \times \langle u, v, 0 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ u & v & 0 \end{vmatrix} = \langle 0, 0, v_x - u_y \rangle$. So we can also perceive $\text{curl}\,\mathbf{U}$ as a scalar even though it is supposed to be a vector parallel to z-axis.
- From now on, let us pretend/define $\text{curl}\,\mathbf{U} = v_x - u_y$ as a scalar for a 2D vector field.
  - $\text{curl}\,\mathbf{U} = v_x - u_y > 0$ means rotation counterclockwise.
  - $\text{curl}\,\mathbf{U} = v_x - u_y < 0$ means rotation clockwise.
  - $\text{curl}\,\mathbf{U} = v_x - u_y = 0$ means no rotation.

Example 1: $\mathbf{U}(x, y) = \langle x, y \rangle$, $\quad \text{div}\, \mathbf{U} = 2$

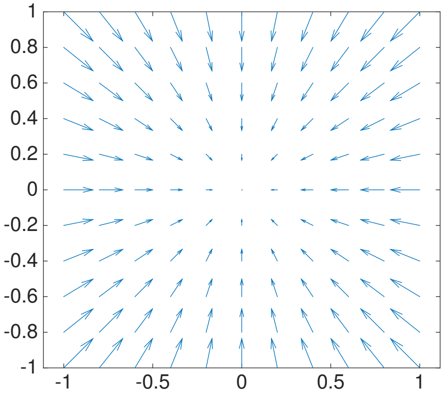
Derivative is a local operator. Notice that $\text{div}\, \mathbf{U}$ is a function as well. So at some point (x_0, y_0), imagine a circle centered at this point with a very small radius (you can let the radius go to zero if that does not confuse you), then $\text{div}\, \mathbf{U}(x_0, y_0) > 0$ simply means that more water molecules are flowing out of this circle than those flowing in. We need Divergence/Gauss Theorem to understand why it is true.



Since $\text{div}\, \mathbf{U} = 2$ holds everywhere, the flow is expanding at the same rate everywhere (not just at the origin).

Example 2: $\mathbf{U}(x, y) = \langle -x, -y \rangle, \qquad \operatorname{div} \mathbf{U} = -2$

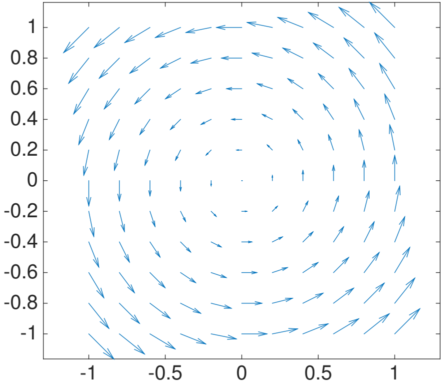
At some point (x_0, y_0), imagine a circle centered at this point with a very small radius, then $\operatorname{div} \mathbf{U}(x_0, y_0) < 0$ simply means that less water molecules are flowing out of this circle than those flowing in.



Since $\operatorname{div} \mathbf{U} = -2$ holds everywhere, the flow is compressing at the same rate everywhere (not just at the origin).

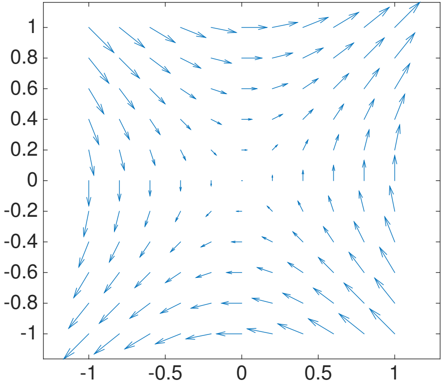
Example 3: $\mathbf{U}(x, y) = \langle -y, x \rangle$, $\quad \operatorname{div} \mathbf{U} = 0 \quad \operatorname{curl} \mathbf{U} = 2$

The flow is incompressible because of zero divergence. The flow is rotating counterclockwise around the origin but that's NOT what $\operatorname{curl} \mathbf{U} = 2$ really means.



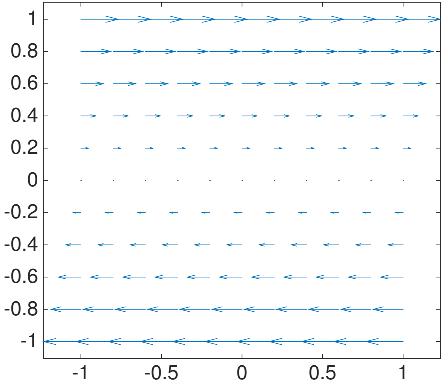
Example 4: $\mathbf{U}(x, y) = \langle y, x \rangle$, $\quad \operatorname{div} \mathbf{U} = 0 \quad \operatorname{curl} \mathbf{U} = 0$

The flow is incompressible and there is no "rotation".



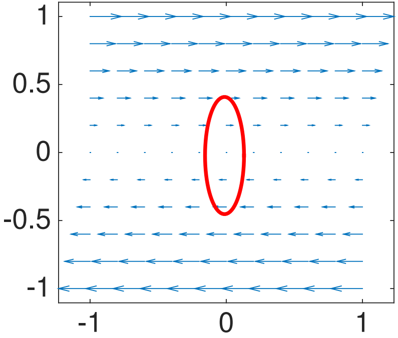
Example 5: $\mathbf{U}(x, y) = \langle y, 0 \rangle$, $\quad$ div $\mathbf{U} = 0$ $\quad$ curl $\mathbf{U} = -1$

The flow is incompressible. The flow is moving horizontally but we expect local clockwise rotation in this flow.



Example 5: $\mathbf{U}(x, y) = \langle y, 0 \rangle$, $\quad \operatorname{div} \mathbf{U} = 0 \quad \operatorname{curl} \mathbf{U} = -1$

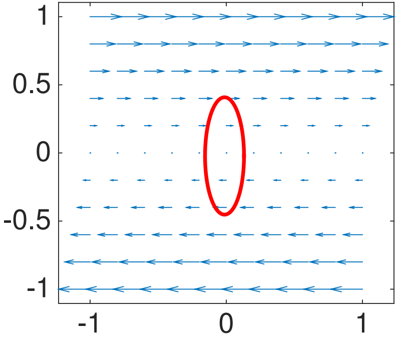
Meaning of local rotation: take the point $(x_0, y_0) = (0, 0)$ as an example, imagine a small boat (size goes to zero if it's OK with you) located at that point. In the figure, the red ellipse represents an exaggerated small boat at $(0, 0)$. What would happen to this boat and why?



Physical Meaning of Curl

Image a small ball with rough surface in a 2D/3D river. The ball will rotate/spin and the rotation axis is the curl vector. The angular speed is $\frac{1}{2}$ of the magnitude of the curl vector.

$$\mathbf{U}(x, y) = \langle y, 0 \rangle, \quad \operatorname{curl} \mathbf{U} = \langle 0, 0, -1 \rangle$$

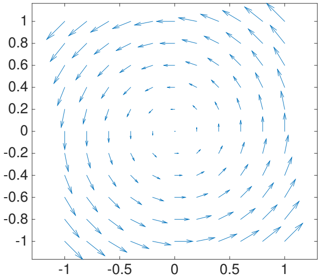


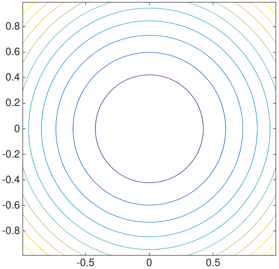
Stream function and stream lines

Example 3: $\mathbf{U}(x, y) = \langle -y, x \rangle$, $\quad \operatorname{div} \mathbf{U} = 0 \quad \operatorname{curl} \mathbf{U} = 2$

Stream lines of a 2D vector field $\langle u, v \rangle$ are the curves with tangent vectors $\langle u, v \rangle$. Or you can think of it as the trajectory of any water molecule.

Image the small ball/boat as a roller coaster ride, then the ride will go along the stream lines. If curl is zero, then it is just a normal train ride along the curve. If curl is positive, then the ride also spins counterclockwise.

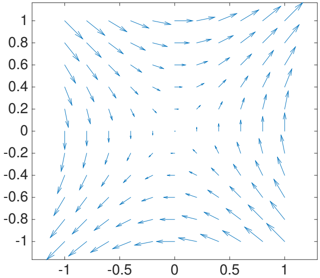


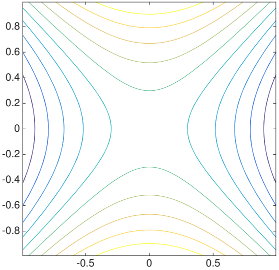


Stream lines are level curves $f(x, y) = C$ where the stream function f satisfies $-f_y = u$ and $f_x = v$ (why is this?). For this example, $f(x, y) = x^2 + y^2$.

Revisit Example 4: $\mathbf{U}(x, y) = \langle y, x \rangle, \quad \text{div}\,\mathbf{U} = 0 \quad \text{curl}\,\mathbf{U} = 0$

This is just a normal train ride along the stream lines.





Stream function is $f(x, y) = x^2 - y^2$. $-f_y = u$ and $f_x = v$.

An incompressible flow simulation

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