Homework 3

Due on Feb 5 in class.

1. A function $f(x, y)$ is called *continuously differentiable* at a point (x_0, y_0) if both of its partial derivatives exist and are continuous at (x_0, y_0). We also call them C^1 functions, i.e., f being a C^1 function means $f(x, y)$ is continuously differentiable. Decide whether the following functions are C^1 at all points $(x, y) \in \mathbb{R}^2$:

   (a)
   $$f(x,y) = \begin{cases} 0, & \text{if } (x,y) = (0,0), \\ \dfrac{2xy}{(x^2+y^2)^2}, & \text{if } (x,y) \neq (0,0). \end{cases}$$

   (b)
   $$f(x,y) = \begin{cases} 0, & \text{if } (x,y) = (0,0), \\ \dfrac{xy}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq (0,0). \end{cases}$$

   Hint: first find the partial derivatives functions $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ which are also piece-wisely defined. Then check whether they are continuous at all points in $\mathbb{R}^2$.

2. Let $f(x, y, z)$ be a scalar-valued differentiable function. Making the substitution
   $$x = \rho \cos\theta \sin\phi, \;\; y = \rho \sin\theta \sin\phi, \;\; z = \rho\cos\phi$$
   into $f(x, y, z)$, compute $\frac{\partial f}{\partial \rho}, \frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \phi}$ in terms of $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$.

3. Compute the directional derivatives of the following functions $f(\mathbf{x})$ along the unit vector parallel to a given vector $\mathbf{v}$ at the point $\mathbf{x}_0$:

   (a)
   $$f(x,y) = x^y, \;\; \mathbf{x}_0 = (e, e), \;\; \mathbf{v} = (5, 12).$$

   (b)
   $$f(x,y,z) = e^x + yz, \mathbf{x}_0 = (1,1,1), \;\; \mathbf{v} = (1,-1,1).$$

4. Find the equation for the plane tangent to the surface $x^2+2y^2+3xz = 10$ at $(1, 2, \frac{1}{3})$.

5. Find the gradient of the function $f(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$.