Poincaré's Lemma for 2-forms

Theorem (Poincaré's Lemma). *For a 2-form with C^1 coefficients on $\mathbb{R}^3$:*

$$\omega = f(x,y,z)dy \wedge dz + g(x,y,z)dz \wedge dx + h(x,y,z)dx \wedge dy,$$

if $d\omega = 0$, then we can find a 1-form

$$\alpha = F(x,y,z)dx + G(x,y,z)dy + H(x,y,z)dz$$

such that $d\alpha = \omega$.

Remark 1. The theorem can be translated to a vector calculus statement: if the vector field $\langle f,g,h\rangle$ satisfies $\nabla \cdot \langle f,g,h\rangle = 0$, then we can find $\langle F,G,H\rangle$ s.t.

$$\nabla \times \langle F,G,H\rangle = \langle f,g,h\rangle.$$

Remark 2. The vector filed $\langle F,G,H\rangle$ is not unique. There are many proofs of this theorem. Different proofs give different vector filed $\langle F,G,H\rangle$.

Proof. **Step I**: split ω into two parts so that

1. the first part does not contain dz,
2. dz appears as the last one in wedge products in the second part.

So we get

$$\omega = \omega_1 + \omega_2$$

with

$$\omega_1 = h(x,y,z)dx \wedge dy, \quad \omega_2 = f(x,y,z)dy \wedge dz - g(x,y,z)dx \wedge dz.$$

Step II: construct a 1-form β by integrating the coefficients in ω_2 w.r.t. z:

$$\beta = \left[\int_0^z f(x,y,t)dt\right] dy + \left[\int_0^z -g(x,y,t)dt\right] dx,$$

then

$$\begin{aligned} d\beta = f(x,y,z)dz \wedge dy + \left(\int_0^z f_x(x,y,t)dt\right) dx \wedge dy \\ - g(x,y,z)dz \wedge dx + \left(\int_0^z -g_y(x,y,t)dt\right) dy \wedge dx, \end{aligned}$$

thus

$$\begin{aligned}\omega_2 + d\beta &= \left(\int_0^z f_x(x,y,t) + g_y(x,y,z)dt\right) dx \wedge dy \\ &= \left(\int_0^z -h_z(x,y,z)dt\right) dx \wedge dy \\ &\quad \text{(because } d\omega = 0 \text{ implies } \quad f_x + g_y + h_z = 0) \\ &= (-h(x,y,z) + h(x,y,0))dx \wedge dy,\end{aligned}$$

so we have

$$\begin{aligned}\omega + d\beta &= \omega_1 + \omega_2 + d\beta \\ &= \omega_1 + (-h(x,y,z) + h(x,y,0))dx \wedge dy \\ &= h(x,y,0)dx \wedge dy.\end{aligned}$$

In other words $\omega + d\beta$ should not contain dz now.

Step III: If necessary, rewrite $\omega + d\beta$ so that dy is the last one in the wedge product:

$$\omega + d\beta = h(x,y,0)dx \wedge dy.$$

Construct another 1-form γ by integrating the coefficient function w.r.t. dy:

$$\gamma = \left(\int_0^y h(x,t,0)dt\right) dx,$$

then

$$d\gamma = h(x,y,0)dy \wedge dx,$$

thus

$$\omega + d\beta + d\gamma = 0.$$

Therefore

$$\omega = -d\beta - d\gamma = d(-\beta - \gamma),$$

thus

$$\begin{aligned}&\alpha = -\beta - \gamma \\ &= -\left(\int_0^z f(x,y,t)dt\right) dy - \left(\int_0^z -g(x,y,t)dt\right) dx - \left(\int_0^y h(x,t,0)dt\right) dx \\ &= \left(\int_0^z g(x,y,t)dt - \int_0^y h(x,t,0)dt\right) dx + \left(-\int_0^z f(x,y,t)dt\right) dy + 0 * dz.\end{aligned}$$

In other words, we get

$$
\begin{aligned}
F &= \int_0^z g(x,y,t)dt - \int_0^y h(x,t,0)dt \\
G &= -\int_0^z f(x,y,t)dt \\
H &= 0.
\end{aligned}
$$

We need to show $\nabla \times \langle F, G, H \rangle = \langle f, g, h \rangle$. Since $\nabla \times \langle F, G, H \rangle = \langle H_y - G_z, F_z - H_x, G_x - F_y \rangle$, we need to show

$$
\begin{aligned}
H_y - G_z &= f \\
F_z - H_x &= g \\
G_x - F_y &= h.
\end{aligned}
$$

We have

$$
\begin{aligned}
H_y - G_z = -G_z = f(x,y,z), \\
F_z - H_x = F_z = g(x,y,z),
\end{aligned}
$$

and

$$
\begin{aligned}
G_x - F_y &= -\int_0^z f_x(x,y,t)dt - \int_0^z g_x(x,y,t)dt + h(x,y,0) \\
&= \int_0^z h_z(x,y,t)dt + h(x,y,0) = h(x,y,z),
\end{aligned}
$$

where we have used the fact $f_x + g_y + h_z = 0$. □

Example 1. Find α s.t. $d\alpha = ydx \wedge dz + zdx \wedge dy$

Solution: $d\alpha = -ydz \wedge dx + zdx \wedge dy$ so

$$f = 0, g = -y, h = z.$$

Thus

$$
\begin{aligned}
F &= \int_0^z g(x,y,t)dt - \int_0^y h(x,t,0)dt = \int_0^z -ydt - \int_0^y 0dt = -yz \\
G &= -\int_0^z f(x,y,t)dt = 0 \\
H &= 0.
\end{aligned}
$$