Today's Plan:

Introduction/Definition of { Embedded Submanifold in a Linear space
Tangent Vector of a manifold

- A real linear space (a.k.a. an abstract vector space over $\mathbb{R}$) is a set $\mathcal{E}$ with addition and scalar multiplication defined and $\mathcal{E}$ is closed under these two operations:

  ① $\forall u, v \in \mathcal{E}$, $u+v \in \mathcal{E}$

  ② $\forall a \in \mathbb{R}$, $\forall u \in \mathcal{E}$, $a \cdot u \in \mathcal{E}$

  Examples: ① $\mathcal{E} = \mathbb{R}^n$

  ② $\mathcal{E} = \mathbb{R}^{n \times p}$

  ③ $\mathcal{E} = L^2(\Omega) = \{f(x) : \int_\Omega |f(x)|^2 dx < \infty\}$

- Let $\mathcal{E}$ be a linear space. Let $\langle u, v \rangle$ be an inner product. With an inner product, we can define distance and open ball, and also open sets.

$\mathcal{E} = \mathbb{R}^{n \times p}$, $\langle U, V \rangle = \mathrm{tr}(V^T U) = \mathrm{tr}(U^T V) = \sum_{i=1}^{n} \sum_{j=1}^{p} U_{ij} V_{ij}$

Frobenius Norm $\|U\|_F = \sqrt{\langle U, U \rangle} = \sqrt{\mathrm{tr}(U^T U)} = \sqrt{\sigma_1^2 + \cdots + \sigma_p^2}$

- Let $\mathcal{E}$ and $\mathcal{F}$ be two linear spaces with inner products defined. Let $U \subset \mathcal{E}$, $V \subset \mathcal{F}$ be open sets.

For a smooth map $F : U \to V$

its differential at $x_0 \in U$ is $DF(x_0) : \mathcal{E} \to \tilde{\mathcal{F}}$ defined by

$$\forall u \in \mathcal{E},\ DF(x_0)[u] = \frac{d}{dt} F[x_0 + tu]\Big|_{t=0} = \lim_{t \to 0} \frac{F(x_0+tu) - F(x_0)}{t}$$

Facts : $DF(x_0)$ is a linear map

Example: ① $\mathcal{E} = \mathbb{R}^3$, $\tilde{\mathcal{F}} = \mathbb{R}$, $F : U \subseteq \mathbb{R}^3 \to V \subseteq \mathbb{R}$

$DF(x_0)(u) = \frac{d}{dt} F(x_0 + tu)$ is the directional derivative.

$DF(x_0)$ is the mapping from u to $DF(x_0)[u]$

② $\mathcal{E} = \mathbb{R}^{n \times p}$, $\tilde{\mathcal{F}} = \mathbb{R}^{p \times p}$ $\quad n > p$

$h : \mathcal{E} \to \tilde{\mathcal{F}}$

$X \longmapsto X^T X - I_p$

$\boxed{X^T}\boxed{X} = \square$

$$Dh(X)[U] = \lim_{t \to 0} \frac{\left([X+tU]^T[X+tU] - I\right) - (X^TX - I)}{t}$$

$$= \lim_{t \to 0} \frac{tU^TX + tX^TU + t^2U^TU}{t}$$

$$= U^TX + X^TU$$

- For a smooth function $f : \mathcal{E} \to \mathbb{R}$, $\langle \nabla f, \vec{u} \rangle = \frac{\partial f}{\partial \vec{u}}$

the Euclidean gradient $\operatorname{grad} f : \mathcal{E} \to \mathcal{E}$ is defined by

$$\langle \operatorname{grad} f(x), v \rangle = Df(x)[v] \qquad \forall x, v \in \mathcal{E}$$

Example: $\mathcal{E}=\mathbb{R}^3$, $\tilde{\mathcal{F}}=\mathbb{R}$, $F: U\subseteq\mathbb{R}^3 \to V\subseteq\mathbb{R}$

$DF(X_0)(u)$ is the directional derivative.

$\Rightarrow$ grad $F(X_0) = \nabla F(X_0)$

Example: $\mathcal{E}=\mathbb{R}^{n\times p}$,

$$f(X) = \langle X, X\rangle = \|X\|_F^2 = \sum_{i=1}^{n}\sum_{j=1}^{p} X_{ij}^2$$

$$Df(X)[V] = \lim_{t\to 0}\frac{f(X+tV)-f(X)}{t}$$

$$= \lim_{t\to 0}\sum_{i=1}^{n}\sum_{j=1}^{p}\frac{|X_{ij}+tV_{ij}|^2-|X_{ij}|^2}{t}$$

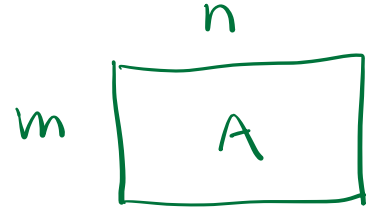
$$= 2\sum_{i=1}^{n}\sum_{j=1}^{p} X_{ij}\cdot V_{ij} = \langle 2X, V\rangle$$

$\boxed{\langle \text{grad} f(X), V\rangle = Df(X)[V]}$ $\Rightarrow$ grad $f(X) = 2X$

- Rank of a linear map

Let $\mathcal{E}$ & $\tilde{\mathcal{F}}$ be finite dimensional linear spaces

For a linear map from $\mathcal{E}$ to $\tilde{\mathcal{F}}$, it can be expressed as $L_A: \mathbb{R}^n \to \mathbb{R}^m$, $x \mapsto Ax$



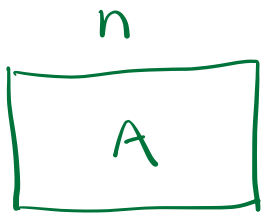
under some bases for $\mathcal{E}$ & $\tilde{\mathcal{F}}$.

The matrix A is different } for different bases.

But Rank(A) is the same

- A surjective linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ has rank m.

$$L_A : \mathbb{R}^n \to \mathbb{R}^m, \quad x \mapsto Ax$$



Surjectivity $\Rightarrow \mathbb{R}^m = \{Ax : x \in \mathbb{R}^n\}$ = Col Space of A

$\Rightarrow$ dim of Col Space is m

- Rank of the differential

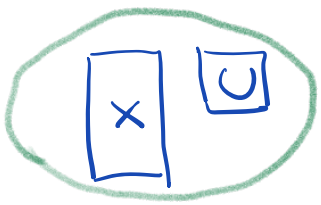
Example: $h : \mathbb{R}^{n\times p} \longrightarrow \mathbb{R}^{p\times p}$ $\quad p \le n$

$$X \longmapsto X^TX - I_p$$

$$Dh(X) : \mathbb{R}^{n\times p} \longrightarrow \mathbb{R}^{p\times p}$$

$$Dh(X)[V] = V^TX + X^TV, \quad \forall V \in \mathbb{R}^{n\times p}$$

$$Dh(X)\left[\tfrac{1}{2}XU\right] = \left(\tfrac{1}{2}XU\right)^TX + X^T\left(\tfrac{1}{2}XU\right)$$



$$= \tfrac{1}{2}U^TX^TX + \tfrac{1}{2}X^TXU = U$$

$U \in \mathbb{R}^{p\times p}$

If $X^TX = I$, $U^T = U$

$\forall U \in S_{p\times p} = \{Y \in \mathbb{R}^{p\times p} : Y^T = Y\}$

$\forall X \in St(n,p) = \{X \in \mathbb{R}^{n\times p} : X^TX = I_p\}$, $p \le n$

$\Rightarrow$ $Dh(X) : \mathbb{R}^{n\times p} \longrightarrow \mathbb{R}^{p\times p}$

can be regarded as a surjective linear map to $S_{p\times p}$

$\Rightarrow$ $Dh(X) : \mathbb{R}^{n\times p} \longrightarrow S_{p\times p} \subseteq \mathbb{R}^{p\times p}$

is surjective if $X \in St(n,p)$

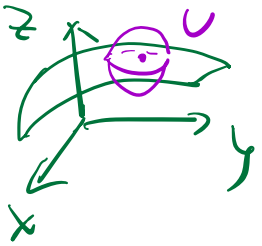
$\Rightarrow$ Rank of $Dh(X)$ is $\frac{p(p+1)}{2}$ if $X \in St(n,p)$

In other words, "Col Space" of $Dh(X)$ is S_{pxp}.

- The generic definition of a manifold and embedded submanifold can be very abstract. To have an easier and quicker understanding of an embedded submanifold in a linear space, we consider the following definition/theorem/fact:

Definition (or Theorem) Let $\mathcal{E}$ be a linear space of dim d and M be an non-empty subset. Let $\tilde{\mathcal{F}}$ be another linear space.

If $\forall X \in M$, there exist an open set U in $\mathcal{E}$, $X \in U$ and a smooth $h: U \to \tilde{\mathcal{F}}$ such that



① $\forall y \in U$, $h(y) = 0 \Leftrightarrow y \in M$

② Rank $[Dh(x)] = k$, $\forall X \in M$

then M is an embedded submanifold of $\mathcal{E}$ of dimension $d-k$

Remark: $m \boxed{A}^{d}$ $\quad L_A: \mathbb{R}^d \to \mathbb{R}^m$, $x \mapsto Ax$

Nullity (A) + Rank $(A) = d$

$Dh(x): \mathbb{R}^d \to \mathbb{R}^k$

Rank $[Dh(x)] = k \Rightarrow \underbrace{\dim(\ker[Dh(x)])}_{\text{dimension of submanifold}} = d-k$

Remark: This is a straightforward extension of definition of surfaces

Example: Any surface S described by $h(x,y,z) = 0$

such as $h(x,y,z) = x^2+y^2-z$

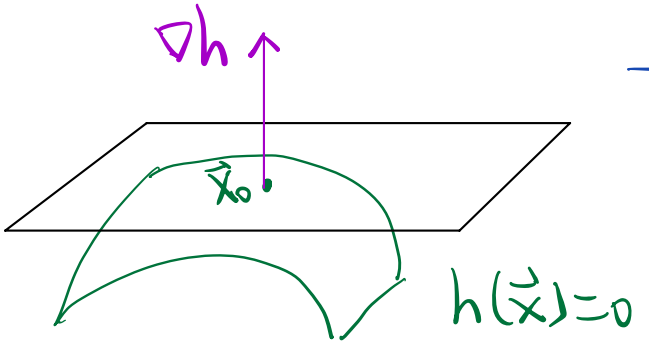
$$Dh(\vec{x})[\vec{u}] = \begin{pmatrix} hx \\ hy \\ hz \end{pmatrix} \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{bmatrix} hx & hy & hz \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$\forall \vec{x} \in S$, $\begin{pmatrix} hx \\ hy \\ hz \end{pmatrix} \neq \vec{0}$, so Rank $[Dh(\vec{x})] = 1$

$Dh(\vec{x}): \mathbb{R}^3 \to \mathbb{R}$ matrix is $[hx \; hy \; hz]$

Kernel or Null space of $Dh(\vec{x}_0)$ is

$$\ker[Dh(\vec{x}_0)] = \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \mathbb{R}^3 : \begin{pmatrix} hx \\ hy \\ hz \end{pmatrix} \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} = 0 \right\}$$



The tangent plane at $\vec{x}_0$ has equation

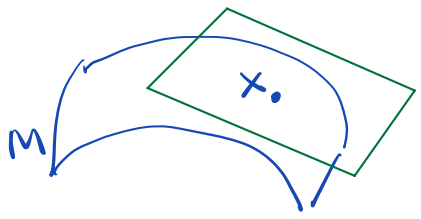
$$\begin{pmatrix} hx(\vec{x}_0) \\ hy(\vec{x}_0) \\ hz(\vec{x}_0) \end{pmatrix} \cdot \begin{pmatrix} u-x_0 \\ v-y_0 \\ w-z_0 \end{pmatrix} = 0$$

So the tangent plane of surfaces can be understood as

$\mathrm{Ker}[Dh(\vec{x}_0)] + \vec{x}_0$ or simply $\mathrm{Ker}[Dh(\vec{x}_0)]$

Definition/Theorem Let M be an embedd submanifold of $\mathcal{E}$.

At $x \in M$, the tangent space is $T_xM = \mathrm{Ker}[Dh(\vec{x})]$



Remark:

① $T_xM = \mathrm{Ker}[Dh(\vec{x})]$ is a linear space

② $\mathrm{Ker}[Dh(\vec{x})] + \vec{x}$ is NOT a linear space

Example: Stiefel Manifold

$$St(n,p) = \{X \in \mathbb{R}^{n\times p} : X^TX = I_p\}, \quad p \le n$$

X

Special Cases:

$p=1$ $\quad St(n,1) = S^{n-1} = \{x \in \mathbb{R}^n : x^Tx = 1\}$

$p=n$ $\quad St(n,n) = O(n) = \{X \in \mathbb{R}^{n\times n}, X^TX = I\}$ Orthogonal Group

Want to
- ① Verify it's an embedded submanifold in $\mathbb{R}^{n\times p}$
- ② Calculate its dimension
- ③ Derive its tangent space

$\mathcal{E} = \mathbb{R}^{n\times p}$, $M = St(n,p) \subset \mathcal{E}$, $\tilde{\mathcal{F}} = S_{p\times p}$

Symmetric matrices

$$h : \mathcal{E} \to \tilde{\mathcal{F}}$$
$$X \mapsto X^TX - I_p$$

$h(X) = 0 \Leftrightarrow X \in St(n,p)$

$$Dh(X): \mathcal{E} \longrightarrow \tilde{\mathcal{F}}$$
$$V \longmapsto V^TX + X^TV$$

At $X \in St(n,p)$, $Dh(X)(\frac{1}{2}XU) = U$, $\forall U \in \tilde{\mathcal{F}}$

$\Rightarrow$ $Dh(X)$ is surjective, $X \in St(n,p)$

$\Rightarrow$ $Dh(X)$ has Rank $\frac{p(p+1)}{2}$ at $X \in St(n,p)$

Also, $h(X) = 0 \Leftrightarrow X \in \mathcal{M}$ $(\ker(h) = \mathcal{M})$

$\Rightarrow$ $\mathcal{M}$ is an embedded submanifold of dim $np - \frac{p(p+1)}{2}$ in $\mathcal{E} = \mathbb{R}^{n\times p}$

(Orthogonal Group $O(n) = \{X \in \mathbb{R}^{n\times n} : X^TX = I\}$ is an embedded submanifold of dim $\frac{n(n-1)}{2}$ in $\mathbb{R}^{n\times n}$)

$$Dh(X): \mathcal{E} \longrightarrow \tilde{\mathcal{F}}$$
$$V \longmapsto V^TX + X^TV$$

$\Rightarrow$ Tangent Space is $T_X\mathcal{M} = \{V \in \mathbb{R}^{n\times p} : V^TX + X^TV = 0\}$

Sphere S^{n-1}, $T_X S^{n-1} = \{V \in \mathbb{R}^n : V^TX + X^TV = 0\}$
$= \{V \in \mathbb{R}^n : V^TX = 0\}$

Remark: For an abstract manifold in a generic definition

① Tangent Space $T_X\mathcal{M}$ is always a linear space of the

same dim as M.

② The map $h(x)$ may not exist.

③ When we have such a map, $\begin{cases} \ker(h) = M \\ \ker(Dh(x)) = T_x M \end{cases}$