Plan
① Review of Linear Algebra
② Surface, Stiefel Manifold, embedded submanifold
③ Riemmanian Metric

Review

I. Linear Algebra

① Rank of a matrix A is equal to

- number of pivots in row echelon form
- dim of row space
- dim of col space
- # of nonzero singular values (including repeated ones)
- # of nonzero eigenvalues if A is { square & diagonalizable

② Rank of a linear mapping $\mathcal{L}: \mathcal{E} \to \tilde{\mathcal{F}}$

Given two bases of $\mathcal{E}$ & $\tilde{\mathcal{F}}$, $\mathcal{L}$ has a matrix representation. The rank $\mathcal{L}$ is defined as rank of its matrix representation

Fact: different bases of $\mathcal{E}$ & $\tilde{\mathcal{F}}$ give { different matrices, same rank

Example: $\mathcal{E} = P_2 = \{ ax^2+bx+c : a,b,c \in \mathbb{R} \}$

$\tilde{\mathcal{F}} = P_1 = \{ ex+d : e,d \in \mathbb{R} \}$

Under natural bases, $p(x) = ax^2+bx+c \sim \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Consider $\mathcal{L} : \mathcal{E} \longrightarrow \tilde{\mathcal{F}}$

$p(x) \longmapsto p'(x)$

$ax^2+bx+c \longmapsto 2ax+b$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \longmapsto \begin{bmatrix} 2a \\ b \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{bmatrix} c \\ b \\ a \end{bmatrix} \longmapsto \begin{bmatrix} b \\ 2a \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix}$$

$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ are matrix representations of $\mathcal{L}$

③ Rank & Nullity

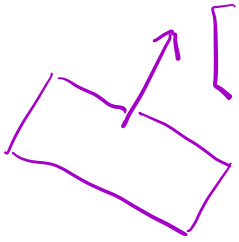
Nullity (A) is dim of $\text{Ker}(A) = \{x : Ax = 0\}$

$$\begin{array}{c} d \\ m \boxed{A} \end{array} \qquad L_A : \mathbb{R}^d \longrightarrow \mathbb{R}^m$$

$$x \longmapsto Ax$$

$\text{Nullity}(A) + \text{Rank}(A) = d \Rightarrow \text{Nullity}(A) = d - \text{Rank}(A)$

Example: $A = [1 \ 2 \ 3]$ $\quad \text{Ker}(A) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : [1 \ 2 \ 3] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \right\}$



$1 \cdot x + 2 \cdot y + 3 \cdot z = 0$ is a plane, and also $\text{Ker}(A)$

II. Differential

- Let $\mathcal{E}$ and $\tilde{\mathcal{F}}$ be two linear spaces with inner products defined.

For a smooth map $F: \mathcal{E} \to \tilde{\mathcal{F}}$

its differential at $x_0 \in \mathcal{E}$ is $DF(x_0): \mathcal{E} \to \tilde{\mathcal{F}}$ defined by

$$\forall u \in \mathcal{E},\ DF(x_0)[u] = \frac{d}{dt} F[x_0 + tu]\Big|_{t=0} = \lim_{t\to 0} \frac{F(x_0+tu) - F(x_0)}{t}$$

Facts: $DF(x_0)$ is a linear map

Example: ① $\mathcal{E} = \mathbb{R}^3$, $\tilde{\mathcal{F}} = \mathbb{R}$, $F: \mathbb{R}^3 \to \mathbb{R}$

$DF(x_0)(u) = \frac{d}{dt} F(x_0 + tu)$ is the directional derivative.

$DF(x_0)$ is the mapping from u to $DF(x_0)[u]$

② $\mathcal{E} = \mathbb{R}^{n\times p}$, $\tilde{\mathcal{F}} = \mathbb{R}^{p\times p}$ $\quad n > p$

$h: \mathcal{E} \to \tilde{\mathcal{F}}$

$X \mapsto X^T X - I_p$

$\boxed{X^T}\ \boxed{X} = \square$

$$Dh(X)[U] = \lim_{t\to 0} \frac{\left([X+tU]^T[X+tU] - I\right) - (X^TX - I)}{t}$$

$$= \lim_{t\to 0} \frac{tU^TX + tX^TU + t^2U^TU}{t}$$

$$= U^TX + X^TU$$

III. **gradf is the representation of differential**

- For a smooth function $f: \mathcal{E} \to \mathbb{R}$,

the Euclidean gradient $\operatorname{grad} f : \mathcal{E} \to \mathcal{E}$ is defined by

$$\langle \operatorname{grad} f(x), v \rangle = Df(x)[v] \qquad \forall x, v \in \mathcal{E}$$

Example: $\mathcal{E} = \mathbb{R}^3$, $\mathcal{F} = \mathbb{R}$, $F : \mathbb{R}^3 \to \mathbb{R}$

$DF(x_0)[u]$ is the directional derivative.

$\langle \nabla F, v \rangle = DF(x_0)[v] \Rightarrow \operatorname{grad} F(x_0) = \nabla F(x_0)$

Example: $\mathcal{E} = \mathbb{R}^{n \times p}$,

$$f(X) = \langle X, X \rangle = \|X\|_F^2 = \sum_{i=1}^{n} \sum_{j=1}^{p} X_{ij}^2$$

$$Df(X)[V] = \lim_{t \to 0} \frac{f(X + tV) - f(X)}{t}$$

$$= \lim_{t \to 0} \sum_{i=1}^{n} \sum_{j=1}^{p} \frac{|X_{ij} + tV_{ij}|^2 - |X_{ij}|^2}{t}$$

$$= 2 \sum_{i=1}^{n} \sum_{j=1}^{p} X_{ij} \cdot V_{ij} = \langle 2X, V \rangle$$

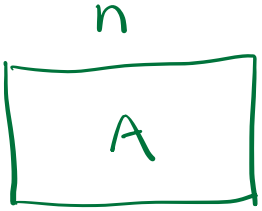
$\boxed{\langle \operatorname{grad} f(X), V \rangle = Df(X)[V]} \Rightarrow \operatorname{grad} f(X) = 2X$

IV. Rank of the differential

① A surjective linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ has rank m.

$L_A : \mathbb{R}^n \to \mathbb{R}^m$

$x \mapsto Ax$



Surjectivity $\Rightarrow \mathbb{R}^m = \{ Ax : x \in \mathbb{R}^n \}$ = Col Space of A

$\Rightarrow$ dim of Col Space is m

(2) Example: $h : \mathbb{R}^{n\times p} \longrightarrow \mathbb{R}^{p\times p}$ $\quad p \le n$

$$X \longmapsto X^T X - I_p$$

$$Dh(X) : \mathbb{R}^{n\times p} \longrightarrow \mathbb{R}^{p\times p}$$

$$Dh(X)[V] = V^T X + X^T V, \quad \forall V \in \mathbb{R}^{n\times p}$$

$$Dh(X)\left[\tfrac{1}{2} X U\right] = \left(\tfrac{1}{2} X U\right)^T X + X^T \left(\tfrac{1}{2} X U\right) \qquad U \in \mathbb{R}^{p\times p}$$

$$= \tfrac{1}{2} U^T X^T X + \tfrac{1}{2} X^T X U \overset{\downarrow}{=} U$$

If $X^T X = I$, $U^T = U$

$\forall U \in S_{p\times p} = \{Y \in \mathbb{R}^{p\times p} : Y^T = Y\}$

$\forall X \in St(n,p) = \{X \in \mathbb{R}^{n\times p} : X^T X = I_p\}$, $p \le n$

$\Rightarrow$ $Dh(X) : \mathbb{R}^{n\times p} \longrightarrow \mathbb{R}^{p\times p}$

can be regarded as a surjective linear map to $S_{p\times p}$

$\Rightarrow$ $Dh(X) : \mathbb{R}^{n\times p} \longrightarrow S_{p\times p} \subseteq \mathbb{R}^{p\times p}$

is surjective if $X \in St(n,p)$

$\Rightarrow$ Rank of $Dh(X)$ is $\dfrac{p(p+1)}{2}$ if $X \in St(n,p)$

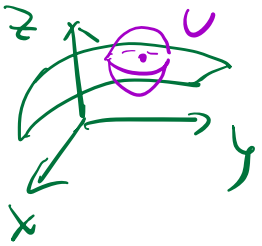
In other words, "Col Space" of $Dh(X)$ is $S_{p\times p}$.

V. Embedded submanifold in $\mathbb{E}$

① **Definition** (or Theorem) Let $\mathcal{E}$ be a linear space of dim d and M be an non-empty subset.

Let $\tilde{\mathcal{F}}$ be another linear space.

If $\forall x \in M$, there exist an open set U in $\mathcal{E}$, $x \in U$



and a smooth $h: U \to \tilde{\mathcal{F}}$ such that

① $\forall y \in U$, $h(y) = 0 \Leftrightarrow y \in M$

② $\text{Rank}[Dh(x)] = k$, $\forall x \in M$

then M is an embedded submanifold of $\mathcal{E}$ of dimension $d-k$

Remark: $m \boxed{A}^{d}$ $\quad L_A: \mathbb{R}^d \to \mathbb{R}^m$, $x \mapsto Ax$

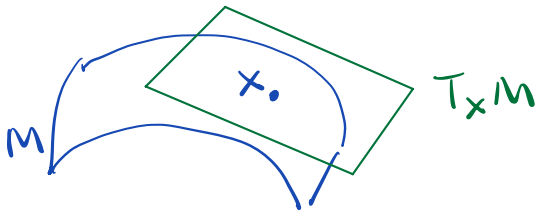
$\text{Nullity}(A) + \text{Rank}(A) = d$

$Dh(x): \mathbb{R}^d \to \mathbb{R}^k$

$\text{Rank}[Dh(x)] = k \Rightarrow \underbrace{\dim(\text{Ker}[Dh(x)])}_{\text{dimension of submanifold}} = d-k$

② **Definition/Theorem** Let M be an embedd submanifold of $\mathcal{E}$.

At $x \in M$, the tangent space is $T_xM = \text{Ker}[Dh(\vec{x})]$



③ Surface Example

Any surface S described by $h(x,y,z)=0$

such as $h(x,y,z)=x^2+y^2-z$

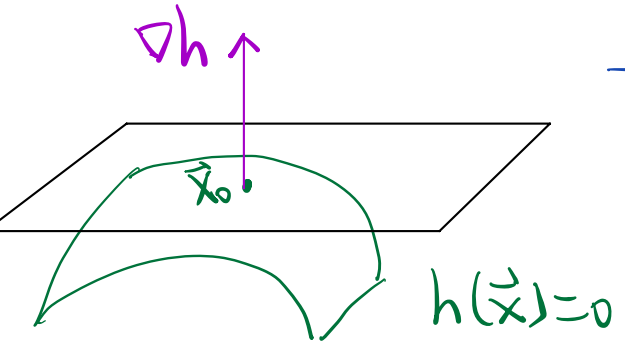
$$Dh(\vec{x})[\vec{u}] = \begin{pmatrix} hx \\ hy \\ hz \end{pmatrix} \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{bmatrix} hx & hy & hz \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$\forall \vec{x} \in S$, $\begin{pmatrix} hx \\ hy \\ hz \end{pmatrix} \neq \vec{0}$, so $\text{Rank}[Dh(\vec{x})] = 1$

$Dh(\vec{x}) : \mathbb{R}^3 \to \mathbb{R}$ matrix is $[hx \; hy \; hz]$

Kernel or Null space of $Dh(\vec{x}_0)$ is

$$\text{Ker}[Dh(\vec{x}_0)] = \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \mathbb{R}^3 : \begin{pmatrix} hx \\ hy \\ hz \end{pmatrix} \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} = 0 \right\}$$



The tangent plane at $\vec{x}_0$ has equation

$$\begin{pmatrix} hx(\vec{x}_0) \\ hy(\vec{x}_0) \\ hz(\vec{x}_0) \end{pmatrix} \cdot \begin{pmatrix} u-x_0 \\ v-y_0 \\ w-z_0 \end{pmatrix} = 0$$

So the tangent plane of surfaces can be understood as

$\text{Ker}[Dh(\vec{x}_0)] + \vec{x}_0$ or simply $\text{Ker}[Dh(\vec{x}_0)]$

④ Compact Stiefel Manifold

$St(n,p) = \{ X \in \mathbb{R}^{n \times p} : X^T X = I_p \}$, $p \leq n$ $\boxed{X}$

Special Cases:

$p=1$ $\quad St(n,1) = S^{n-1} = \{x \in \mathbb{R}^n : x^T x = 1\}$

$p=n$ $\quad St(n,n) = O(n) = \{x \in \mathbb{R}^{n\times n}, X^T X = I\}$ Orthogonal Group

Want to { ① Verify it's an embedded submanifold in $\mathbb{R}^{n\times p}$
② Calculate its dimension
③ Derive its tangent space

$\mathcal{E} = \mathbb{R}^{n\times p}$, $M = St(n,p) \subset \mathcal{E}$, $\tilde{\mathcal{F}} = S_{p\times p}$

Symmetric matrices

$$h : \mathcal{E} \longrightarrow \tilde{\mathcal{F}}, \quad X \longmapsto X^T X - I_p$$

$h(X) = 0 \Leftrightarrow X \in St(n,p)$

$$Dh(X) : \mathcal{E} \longrightarrow \tilde{\mathcal{F}}, \quad V \longmapsto V^T X + X^T V$$

At $X \in St(n,p)$, $Dh(X)(\frac{1}{2}XU) = U$, $\forall U \in \tilde{\mathcal{F}}$

$\Rightarrow Dh(X)$ is surjective, $X \in St(n,p)$

$\Rightarrow Dh(X)$ has Rank $\frac{p(p+1)}{2}$ at $X \in St(n,p)$

Also, $h(X) = 0 \Leftrightarrow X \in M$ $\quad (\ker(h) = M)$

$\Rightarrow M$ is an embedded submanifold of dim $np - \frac{p(p+1)}{2}$ in $\mathcal{E} = \mathbb{R}^{n\times p}$

(Orthogonal Group $O(n) = \{X \in \mathbb{R}^{n\times n} : X^TX = I\}$
is an embedded submanifold of dim $\frac{n(n-1)}{2}$ in $\mathbb{R}^{n\times n}$)

$$Dh(X): \mathcal{E} \longrightarrow \mathcal{F}$$
$$V \longmapsto V^TX + X^TV$$

$\Rightarrow$ Tangent Space is $T_xM = \{V \in \mathbb{R}^{n\times p} : V^TX + X^TV = 0\}$

Sphere S^{n-1}, $T_xS^{n-1} = \{V \in \mathbb{R}^n : V^TX + X^TV = 0\}$
$= \{V \in \mathbb{R}^n : V^TX = 0\}$

Remark: For an abstract manifold in a generic definition

① Tangent Space T_xM is always a linear space of the same dim as M.

② The map $h(x)$ may not exist.

③ When we have such a map, $\begin{cases} \ker(h) = M \\ \ker(Dh(x)) = T_xM \end{cases}$

VI. Definition of Metric and Riemannian Manifold

① If each tangent space T_xM has an inner product, we call it metric, denoted as g_x

i) $\forall u, v \in T_xM$, $g_x(u,v) = g_x(v,u)$

2) $\forall a, b \in \mathbb{R}$, $g_x(au+bv, w) = a g_x(u,w) + b g_x(v,w)$

3) $g_x(u,u) \geq 0$ & $g_x(u,u) = 0 \Leftrightarrow u = 0$

② g_x is smooth w.r.t. x, we say it's Riemmanian

③ A smooth manifold M with a Riemannian metric g is called Riemmanian manifold (M, g)

④ If $M \subset \mathcal{E}$ is an embedded submanifold in $\mathcal{E}$ and g is the inner product in $\mathcal{E}$ (g does not depend on x) then we say M is a Riemmanian submanifold of $\mathcal{E}$.

Example: ① $M = S^{n-1}$, $g_x(u,v) = u^T v$

Then (S^{n-1}, g) is a Riemmanian submanifold of $\mathbb{R}^n$

② $M = S^{n-1}$, $g_x(u,v) = u^T(xx^T + I)v$

(S^{n-1}, g) is a Riemmanian manifold

Layers of concepts for an abstract set M

① A not too crazy set (second countable Hausdorff)

② A topology (has a notion of open sets)

③ A topological manifold (locally Euclidean)

④ A smooth manifold (transition map is smooth)

⑤ A Riemannian manifold (M, g)

⑥ A pseudo-Riemmanian manifold (M, g)

pseudo-Riemmanian metric, denoted as g_x

1) $\forall u, v \in T_x M$, $g_x(u, v) = g_x(v, u)$

2) $\forall a, b \in \mathbb{R}$, $g_x(au+bv, w) = a g_x(u, w) + b g_x(v, w)$

Remark 3) $g_x(u, u) \geq 0$ & $g_x(u, u) = 0 \Leftrightarrow u = 0$

Example: General Relativity

Lorentzian Manifold of Space Time

$$g\left[\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ t_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ t_2 \end{pmatrix}\right] = x_1x_2 + y_1y_2 + z_1z_2 - t_1t_2$$