Assume $f(x): \mathbb{R}^n \to \mathbb{R}$ is convex and has a minimizer x_*.

Then convexity $\Rightarrow \begin{cases} ① \ f(x) \text{ is continuous} \\ ② \ \partial f(x) \text{ is nonempty at any } x. \end{cases}$

① Consider $f(x): \mathbb{R} \to \mathbb{R}$.

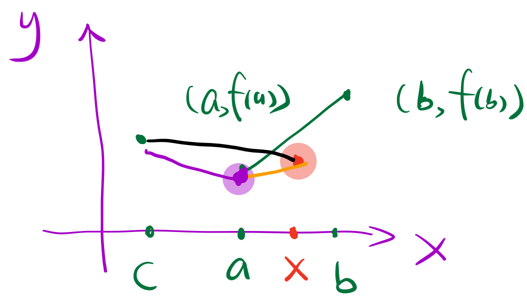
Show convexity $\Rightarrow$ continuity

Proof: For any a, want to show

$$|f(x) - f(a)| \to 0 \quad \text{as } x \to a$$

Two cases:

1) $x > a$



$f(x)$ is below green line segment

$f(a)$ is below black line segment

$$\Rightarrow \begin{cases} \dfrac{f(x)-f(a)}{x-a} \leq \dfrac{f(b)-f(a)}{b-a} \\[2ex] \dfrac{f(x)-f(a)}{x-a} \geq \dfrac{f(c)-f(a)}{c-a} \end{cases}$$

$$(x-a)\frac{f(c)-f(a)}{c-a} \leq f(x) - f(a) \leq (x-a)\frac{f(b)-f(a)}{b-a}$$

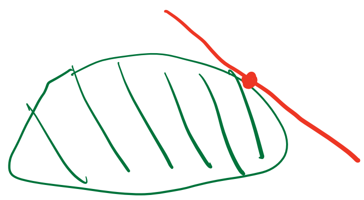
$$x - a \to 0 \Rightarrow f(x) \to f(a).$$

2) $x < a$ is similar

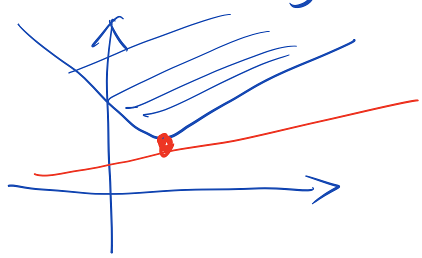
② Show convexity $\Rightarrow$ subgradient exists at any x.

Sketchy proof: 1) The epigraph of $f(x)$ is a convex set

2) Any convex set has a supporting plane



3) The supporting plane of the epigraph is a sub-tangent line



Now consider $\min_{x \in \mathbb{R}^n} f(x)$

where $\begin{cases} f(x) \text{ is convex on } \mathbb{R}^n \\ f(x) \geq f(x_*) \end{cases}$

$0 \in \partial f(x_*)$

Two simple algorithms

$\frac{d}{dt} X(t) = f[X(t)]$

① Subgradient Method

$X_{k+1} = X_k - \eta_k g_k, \quad g_k \in \partial f(X_k)$

② Proximal Point Method

$$x_{k+1} = x_k - \eta_k g_{k+1}, \quad g_{k+1} \in \partial f(x_{k+1})$$

$$\iff (I + \eta_k \partial f)(x_{k+1}) = x_k$$

$$\iff x_{k+1} = (I + \eta_k \partial f)^{-1}(x_k)$$

$$\iff x_{k+1} = \mathrm{Prox}_f^{\eta_k}(x_k)$$

$$\mathrm{Prox}_f^{\gamma}(x) = \arg\min_u \left[\gamma f(u) + \tfrac{1}{2}\|u - x\|^2\right]$$

No closed formula for Proximal Operator in general

Example: $f(x) = \|x\|_1$

$$\mathrm{Prox}_f^{\gamma}(x)_i = \begin{cases} x_i - \gamma, & \text{if } x_i > \gamma \\ x_i + \gamma, & \text{if } x_i < -\gamma \\ 0, & \text{if } x_i \in [-\gamma, \gamma] \end{cases}$$

③ Proximal Gradient for Composite Optimization

$$\min_x [f(x) + g(x)]$$

$f(x)$ is nonsmooth
$g(x)$ is smooth

$$x_{k+1} = (I + \eta_k \partial f)^{-1}(x_k - \eta_k \nabla g(x_k))$$

This is also called

$$\min_x \frac{\lambda}{2}\|Ax - b\|^2 + \|x\|_1 + \frac{\mu}{2}\|x\|^2$$

1) Forward - Backward splitting

2) Proximal Gradient $\frac{d}{dt}x = f(x) + g(x)$

3) For ODE, this is Implicit-Explicit (IMEX) method

Convergence Rate for nonsmooth problems

	Convexity	Strong Convexity
Subgradient Method	$O(\frac{1}{\sqrt{k}})$	$O(\frac{1}{k})$
Proximal Point Method	$O(\frac{1}{k})$	$O((1-\mu)^k)$
Proximal Gradient	$O(\frac{1}{k})$	$O((1-\mu)^k)$
Accelerated Proximal Gradient	$O(\frac{1}{k^2})$	$O((1-\sqrt{\mu})^k)$

Convergence Rate for Smooth Convex Problems

$\downarrow$

$\nabla f(x)$ is L-cont.

	Convex	Strongly Convex
Gradient Descent	$O(\frac{1}{k})$	$\frac{L}{2}\left(\frac{L/\mu - 1}{L/\mu - 1}\right)^k$
Accelerated Gradient Method	$O(\frac{1}{k^2})$	

---

① Subgradient Method

$$x_{k+1} = x_k - \eta_k \frac{g_k}{\|g_k\|}, \quad g_k \in \partial f(x_k)$$

$$\|x_{k+1} - x_*\|^2 = \left\| x_k - x_* - \eta_k \frac{g_k}{\|g_k\|} \right\|^2$$

$$= \|x_k - x_*\|^2 - 2\eta_k \frac{1}{\|g_k\|} \langle x_k - x_*, g_k \rangle + \eta_k^2$$

$$\leq \|x_k - x_*\|^2 - 2\eta_k \frac{1}{\|g_k\|} [f(x_k) - f(x_*)] + \eta_k^2$$

$$f(x) \geq f(x_k) + \langle g_k, x - x_k \rangle$$

Assume $\|g_k\| \leq M$, let $f(\bar{x}_k) = \min_{1 \leq i \leq k} f(x_k)$

Sum it for $1, 2, \cdots, n$

$$\Rightarrow \|x_{n+1} - x_*\|^2 \leq \|x_0 - x_*\|^2 - \frac{2}{M}\left(\sum_{k=0}^{n} \eta_k\right)[f(\bar{x}_n) - f(x_*)] + \sum_{k=0}^{n} \eta_k^2$$

$$\Rightarrow f(\bar{x}_n) - f(x_*) \leq M \frac{\sum_{k=0}^{n} \eta_k^2 + \|x_0 - x_*\|^2}{2\sum_{k=0}^{n} \eta_k}$$

1) So for convergence, we can use $\begin{cases} \sum_{k=0}^{\infty} \eta_k = +\infty \\ \sum_{k=0}^{\infty} \eta_k^2 < +\infty \end{cases}$ $\eta_k = \frac{1}{k}$

2) Optimal strategy : $\eta_k = \frac{C}{\sqrt{n+1}}$ for $k = 0, \cdots, n$

$$\Rightarrow \quad f(\bar{x}_n) - f(x_*) \leq M \frac{C \sum_{k=0}^{n} \frac{1}{n+1} + \|x_0 - x_*\|^2}{2C \sum_{k=0}^{n} \frac{1}{\sqrt{n+1}}}$$

$$M = \max_{0 \leq k \leq n} \|g_k\|$$

$$= M \frac{C + \|x_0 - x_*\|^2}{2C\sqrt{n+1}}$$

3) Polyak's step size $\eta_k = \dfrac{f(x_k) - f(x_*)}{\|g_k\|}$

$$\|x_{k+1} - x_*\|^2 \leq \|x_k - x_*\|^2 - 2\eta_k \frac{1}{\|g_k\|}[f(x_k) - f(x_*)] + \eta_k^2$$

$$= \|x_k - x_*\|^2 - \frac{|f(x_k) - f(x_*)|^2}{\|g_k\|^2}$$

$$\leq \|x_k - x_*\|^2 - \frac{|f(x_k) - f(x_*)|^2}{M^2}$$

Sum it for $k = 0, \cdots, n$

$$\frac{1}{M^2} \sum_{k=0}^{n} |f(x_k) - f(x_*)|^2 \leq \|x_0 - x_*\|^2 - \|x_{n+1} - x_*\|^2$$

$$\leq \|x_0 - x_*\|^2$$

$$(n+1)|f(\bar{x}_k) - f(x_*)|^2 \leq \sum_{k=0}^{n} |f(x_k) - f(x_*)|^2$$

$$\Rightarrow \quad f(\bar{x}_k) - f(x_*) \leq \frac{M}{\sqrt{n+1}} \|x_0 - x_*\|$$

$$M = \max_{0 \leq k \leq n} \|g_k\|$$

Now assume strong convexity with $\mu > 0$

$$\|x_{k+1} - x_*\|^2 = \left\| x_k - x_* - \eta_k \frac{g_k}{\|g_k\|} \right\|^2$$

$$= \|x_k - x_*\|^2 - 2\eta_k \frac{1}{\|g_k\|} \langle x_k - x_*, g_k \rangle + \eta_k^2$$

$$f(x) \geq f(x_k) + \langle g_k, x - x_k \rangle + \frac{\mu}{2}\|x - x_k\|^2$$

$$f(x_*) \geq f(x_k) + \langle g_k, x_* - x_k \rangle + \frac{\mu}{2}\|x_* - x_k\|^2$$

$$\Rightarrow -\langle g_k, x_k - x_* \rangle \leq f(x_*) - f(x_k) - \frac{\mu}{2}\|x_* - x_k\|^2$$

$$\leq \left(1 - 2\eta_k \frac{1}{\|g_k\|} \frac{\mu}{2}\right)\|x_k - x_*\|^2$$

$$- 2\eta_k \frac{1}{\|g_k\|}\left[f(x_k) - f(x_*)\right] + \eta_k^2$$

$$\Rightarrow f(x_k) - f(x_*) \leq \left(\frac{\|g_k\|}{2\eta_k} - \frac{\mu}{2}\right)\|x_k - x_*\|^2 - \frac{\|g_k\|}{2\eta_k}\|x_{k+1} - x_*\|^2 + \frac{\|g_k\|}{2}\eta_k$$

$$\eta_k = \frac{2}{\mu(k+1)} \cdot \|g_k\|$$

$$= \frac{\mu(k-1)}{4}\|x_k - x_*\|^2 - \frac{\mu(k+1)}{4}\|x_{k+1} - x_*\|^2 + \frac{1}{\mu(k+1)}\|g_k\|^2$$

$$k\left[f(x_k) - f(x_*)\right] \leq k\frac{\mu(k-1)}{4}\|x_k - x_*\|^2 - \frac{k\mu(k+1)}{4}\|x_{k+1} - x_*\|^2 + k\frac{1}{\mu(k+1)}\|g_k\|^2$$

Sum it for $k = 0, 1, \cdots, n$

$$\sum_{k=0}^{n} k\left[f(x_k) - f(x_*)\right] \leq -\frac{\mu}{4} n(n+1) \|x_{n+1} - x_*\|^2 + \frac{M^2}{\mu} \sum_{k=0}^{n} \frac{k}{k+1}$$

$$\leq \frac{M^2 n}{\mu}$$

$$\Rightarrow \underbrace{\left(\sum_{k=0}^{n} k\right)}_{\frac{n(n+1)}{2}} \left[f(\bar{x}_n) - f(x_*)\right] \leq \frac{M^2 n}{\mu}$$

$$\Rightarrow \quad f(\bar{x}_n) - f(x_*) \leq \frac{2M^2}{\mu(n+1)}$$

$$\frac{\mu}{2} \|\bar{x}_k - x_*\|^2 \leq f(\bar{x}_k) - f(x_*)$$

why?

$$\Rightarrow \quad \|\bar{x}_k - x_*\| \leq \frac{2M}{\mu\sqrt{k+1}}$$