Review

$\boxed{\text{Def}}$ An operator $T: \mathbb{R}^n \to \mathbb{R}^n$ is called

1) a contraction if $\|T(x) - T(y)\| \leq c\,\|x-y\|$

$0 < c < 1$

2) firmly nonexpansive if

$$\|T(x) - T(y)\|^2 \leq \langle T(x) - T(y), x-y \rangle$$

3) nonexpansive if $\|T(x) - T(y)\| \leq \|x-y\|$

$\boxed{\text{Theorem}}$ ① T is firmly nonexpansive

$\Leftrightarrow$ $2T - I$ is nonexpansive

② T is nonexpansive

$\Leftrightarrow$ $\frac{T+I}{2}$ is firmly nonexpansive

Example: ① $f(x)$ is convex $\Rightarrow$

$$\|\text{Prox}_f^\eta(x) - \text{Prox}_f^\eta(y)\|^2 \leq \langle \text{Prox}_f^\eta(x) - \text{Prox}_f^\eta(y), x-y \rangle$$

② $f(x)$ is strongly convex with $\mu > 0$ $\Rightarrow$

$$(1+\mu\eta)\|\text{Prox}_f^\eta(x) - \text{Prox}_f^\eta(y)\|^2 \leq \langle \text{Prox}_f^\eta(x) - \text{Prox}_f^\eta(y), x-y \rangle$$

$$\Rightarrow \|\text{Prox}_f^\eta(x) - \text{Prox}_f^\eta(y)\| \leq \frac{1}{1+\mu\eta}\|x-y\|$$

Theorem (Browder–Göhde–Kirk)

$T: \mathbb{R}^n \to \mathbb{R}^n$ is nonexpansive $\Rightarrow$ T has at least one fixed point.

$T(x_*) = x_*$.

$x_{k+1} = T(x_k)$ may not converge to x_*

Example: $T(x) = -x$ $\quad x_* = 0$

Theorem If $T: \mathbb{R}^n \to \mathbb{R}^n$ is nonexpansive, then

$$x_{k+1} = \theta x_k + (1-\theta) T(x_k), \quad 0 < \theta < 1$$

converges to one fixed point of $T(x)$.

Consider $\min\limits_x f(x) + g(x)$ $\quad$ f & g are convex nonsmooth

$$\min_x \underbrace{\|x\|_1}_{f(x)} + \underbrace{\iota_{\{Ax=b\}}(x)}_{g(x)}$$

$$\boxed{\ A\ }\boxed{x} = \boxed{b}$$

$$\mathrm{Prox}^\eta_f(x)_i = \begin{cases} x_i - \eta, & x_i > \eta \\ x_i + \eta, & x_i < -\eta \\ 0, & x_i \in [-\eta, \eta] \end{cases}$$

$$\mathrm{Prox}^\eta_g(x) = x + A^T (AA^T)^{-1}(b - Ax)$$

Splitting for finding $0 \in \partial f(x_*) + \partial g(x_*)$

I. $T(x) = \mathrm{Prox}_f\left[\mathrm{Prox}^\eta_g(x)\right]$ is nonexpansive,

so T has a fixed point, which is NOT x_*!

$$0 \in \partial f(x_*) + \partial g(x_*) \Leftrightarrow 0 \in -\eta \partial f(x_*) - \eta \partial g(x_*)$$
$$\Leftrightarrow 0 \in [x_* - \eta \partial f(x_*)] - [x_* + \eta \partial g(x_*)]$$
$$\Leftrightarrow [I + \eta \partial g](x_*) \in [I - \eta \partial f](x_*)$$

$x_{k+1} = \theta x_k + (1-\theta) \text{Prox}_f [\text{Prox}_g^\eta (x_k)]$ converges, but not to x_*!

II. Need a convergent iteration to x_*

Douglas-Rachford Splitting (Lions & Mercier 1979)

$f(x)$ is convex $\Rightarrow$ Prox_f^η is firmly nonexpansive

$\Rightarrow R_f = 2\text{Prox}_f^\eta - I$ is nonexpansive

So $T = R_f R_g$ is nonexpansive

$\Rightarrow S = \frac{T+I}{2}$ is firmly nonexpansive

Douglas-Rachford Splitting

$$y_{k+1} = \frac{I + R_f R_g}{2}(y_k) \qquad \theta = \frac{1}{2}$$
$$= \theta y_k + (1-\theta) R_f R_g (y_k)$$

$x_{k+1} = \theta x_k + (1-\theta) T(x_k)$ converges if T is nonexpansive

$0 < \theta < 1$

$y_{k+1} = \frac{I+R_fR_g}{2}(y_k)$ converges to a fixed point of $S = \frac{I+R_fR_g}{2}$ which is NOT x_* !

$$y_* = \frac{I+R_fR_g}{2}(y_*)$$

$$\Leftrightarrow\quad 2y_* = y_* + R_fR_g(y_*)$$

$$\Leftrightarrow\quad y_* = R_f\left[R_g(y_*)\right]$$

$$= 2\,\mathrm{Prox}_f^{\eta}\left[R_g(y_*)\right] - R_g(y_*)$$

$$= 2\,\mathrm{Prox}_f^{\eta}\left[R_g(y_*)\right] - 2\,\mathrm{Prox}_g^{\eta}(y_*) + y_*$$

$$\Leftrightarrow\quad \underbrace{\mathrm{Prox}_g^{\eta}(y_*)}_{z} = \mathrm{Prox}_f^{\eta}\big[\underbrace{R_g(y_*)}_{2\,\mathrm{Prox}_g^{\eta}(y_*)-y_* = 2z-y_*}\big]$$

$(I+\eta\partial g)(z) = y_*$

$$\Leftrightarrow\quad z = (I+\eta\partial f)^{-1}\left[2z-(I+\eta\partial g)(z)\right]$$

$$= (I+\eta\partial f)^{-1}\left[(I-\eta\partial g)(z)\right]$$

$$\Leftrightarrow\quad 0 \in \partial f(z) + \partial g(z)$$

$\boxed{\text{Theorem}}$ (Douglas-Rachford splitting)

Assume ① $f(x)$ and $g(x)$ are convex,

② $f(x)+g(x)$ has a minizer

then the iteration

$$\begin{cases} y_{k+1} = \dfrac{I+R_f R_g}{2}(y_k) \\ x_{k+1} = \mathrm{Prox}_g^{\eta}(y_{k+1}) \end{cases}$$ converges, $\forall \eta > 0$

and $\{x_k\}$ converges to one minimizer of $f(x)+g(x)$

Remark: $\{y_k\}$ is an auxiliary variable

Even if x_* is unique, y_* may be non-unique.

Remark: $$y_{k+1} = \frac{I+R_f R_g}{2}(y_k)$$

$$= \frac{1}{2} y_k + \frac{1}{2}\Big[2\mathrm{Prox}_f\left[R_g(y_k)\right] - R_g(y_k)\Big]$$

$$= \mathrm{Prox}_f\left[2\mathrm{Prox}_g(y_k) - y_k\right] - \mathrm{Prox}_g(y_k) + y_k$$

$$= \mathrm{Prox}_f\left[2x_k - y_k\right] - x_k + y_k$$

Example: $$\min_x \|x\|_1 + i_{\{Ax=b\}}(x) \iff \min_x \|x\|_1 \quad \text{s.t. } Ax=b \qquad \boxed{A}$$

$f(x) + g(x)$

$$\mathrm{Prox}_f^{\eta} = S_\eta(x) \qquad S_\eta(x)_i = \begin{cases} x_i - \eta, & x_i > \eta \\ x_i + \eta, & x_i < -\eta \\ 0, & x_i \in [-\eta, \eta] \end{cases}$$

$$\mathrm{Prox}_g^{\eta} = P(x)$$

$$= x + A^T(AA^T)^{-1}(b - Ax)$$

$$\begin{cases} x_k = \mathrm{Prox}_g^{\eta}(y_k) = y_k + A^T(AA^T)^{-1}(b - Ay_k) \\ y_{k+1} = \mathrm{Prox}_f[2x_k - y_k] - x_k + y_k \\ \quad\;\; = S_\eta[2x_k - y_k] - x_k + y_k \end{cases}$$

III. $f(x)$ and $g(x)$ are convex

$T = R_f R_g(x)$ is non-expansive

$\Rightarrow y_{k+1} = (1-\theta) y_k + \theta R_f R_g(y_k)$, $\theta \in (0,1)$

converges to fixed point y_*

$(T(y_*) = y_*)$

$\theta = \frac{1}{2}\lambda$

$$\Rightarrow y_{k+1} = (1 - \tfrac{\lambda}{2}) y_k + \tfrac{\lambda}{2} R_f R_g(y_k)$$

$$= (1-\lambda) y_k + \underbrace{\left[\tfrac{\lambda}{2} y_k + \tfrac{\lambda}{2} R_f R_g(y_k)\right]}_{(1-\lambda)I + \lambda \frac{I + R_f R_g}{2}}, \quad \lambda \in (0, 2)$$

$\boxed{\text{Theorem}}$ (Generalized Douglas-Rackford, 1992)

Assume ① $f(x)$ and $g(x)$ are convex,

② $f(x) + g(x)$ has a minizer

then the iteration

$$\begin{cases} y_{k+1} = (1-\lambda) y_k + \lambda \dfrac{I + R_f R_g}{2} (y_k) \\ x_{k+1} = \mathrm{Prox}_g^{\eta}(y_{k+1}) \end{cases}$$

converges, $\forall \eta > 0$, $\lambda \in (0,2)$

and $\{x_k\}$ converges to one minimizer of $f(x) + g(x)$

Remark: 1) λ is called Relaxation parameter

2) each λ gives a different algorithm

3) $\lambda = 1$ is Douglas-Rachford

4) $\lambda = 2$: $y_{k+1} = R_f R_g (y_k)$

is Peaceman-Rachford

which may not converge!

IV. Peaceman-Rachford

① The fixed point iteration for convex $f(x)$ & $g(x)$

$y_{k+1} = R_f R_g (y_k)$ converges to y_*

if either $f(x)$ or $g(x)$ is strongly convex.

② $\begin{cases} y_{k+1} = R_f R_g (y_k) \\ x_{k+1} = \mathrm{Prox}_g^{\eta}(y_{k+1}) \end{cases}$ for convex $f(x)$ & $g(x)$

$g(x)$ is strongly convex $\Rightarrow \{x_k\} \to x_*$