MA/CS 615 Spring 2019 Homework #1

Due before class starts on Jan. 29th. Late homework will not be given any credit. Collaboration is OK but not encouraged. Indicate on your report whether you have collaborated with others and whom you have collaborated with.

1. (20 pts)

(a) (10 pts) Consider the $n \times n$ matrix

$$T(\alpha) = \begin{pmatrix} \alpha & -1 & & & & \\ -1 & \alpha & -1 & & & \\ & -1 & \alpha & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & \alpha & -1 \\ & & & & -1 & \alpha \end{pmatrix}.$$

Find its eigenvalues and a complete set of eigenvectors (meaning that they can form an invertible matrix). For what kind of α, this matrix will become positive definite?

(b) (10 pts) Find the eigenvalues and a complete set of eigenvectors for

$$T_2 = \begin{pmatrix} 2 & -2 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}.$$

2. (20 points) Discuss the local truncation error of the 9-point scheme (read the lecture notes) for solving $-u_{xx}(x, y) - u_{yy}(x, y) = f(x, y)$ for the case $\Delta x \neq \Delta y$. Given the explicit expression of the leading term in the local truncation error.

3. (30 points) Consider the following problem with Dirichlet and Neumann boundary conditions:

$$\begin{cases} -u''(x) = 25\pi^2 \sin(5\pi x) - 6x, & x \in (0, 1), \\ u'(0) = 5\pi, \ u(1) = 1. \end{cases}$$

The exact solution is $u(x) = \sin(5\pi x) + x^3$. Modify the sample code Poisson1D_Dirichlet.m provided on the course webpage to implement the following second order accurate finite difference schemes: on the grid $x_j = (j - 1)h$ with $h = \frac{1}{n}$ using the nonsymmetric matrix

$$T_2 = \begin{pmatrix} 2 & -2 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}.$$

The matrix-vector forms of the scheme is $\frac{1}{h^2}T_2U = \widehat{F}$ where $\widehat{F}$ is the modified right hand side data due to the nonzero boundary conditions. Use eigenvector method to invert the matrix to receive the full credit. Use $inv(S)$ in MATLAB to find S^{-1}. Multiply the eigenvector matrices S and S^{-1} directly. Do not consider the Discrete Cosine Transform.

(a) (5 pts) Find the local truncation error at the left boundary.

(b) (10 pts) Prove that stability of the scheme. Namely, let $A = \frac{1}{h^2}T_2$ then find $\|A^{-1}\|$ and show $\|A^{-1}\|$ is bounded by a constant (be rigorous and specific on what this constant is) as $h \to 0$. **Hint**: You can verify that T_2 is NOT a normal matrix thus you cannot use eigenvalues of T_2 to find $\|T_2^{-1}\|$. It is hard to find singular values of T_2. On the other hand, $T_2 = DT$ where T matrix is symmetric (thus normal) and D is a diagonal matrix. Use eigenvalues of T and the inequality $\|AB\| \leq \|A\|\|B\|$ to prove the stability.

(c) (5 pts) Discuss the convergence of the scheme. Find the convergence order (in what norm?).

(d) (10 pts) **In your report, give explicit expressions for $\widehat{F}$.** As in the sample code, provide the loglog plot of errors in max norm and 2-norm of both schemes with comparison to the second order slope line on 10 different grids: $n = 10, 20, 30, \ldots, 100$. Also plot the numerical solutions V.S. the true solution on the finest grid $n = 100$ with the following range on the axis:

```
axis([0 1 -1 2.3])
```

4. (30 points) Design a second order finite difference scheme for solving

$$-u_{xx}(x, y) - u_{yy}(x, y) = \frac{5}{4}\pi^2 \sin(\frac{1}{2}\pi x)\cos(\pi y)$$

on the rectangle $(x, y) \in (0, 3) \times (0, 1.5)$ with the following boundary conditions:

- Dirichlet b.c. on the left and right boundaries: $u(0, y) = 0$, $u(3, y) = \sin(\frac{3}{2}\pi)\cos(\pi y)$.
- Neumann b.c. on the top and bottom boundaries: $\frac{\partial}{\partial y}u(x, 0) = 0$, $\frac{\partial}{\partial y}u(x, 1.5) = -\pi \sin(\frac{1}{2}\pi x)\sin(\pi y)$.

The exact solution is $u(x, y) = \sin(\frac{1}{2}\pi x)\cos(\pi y)$. Use the eigenvector method to implement the scheme. See Poisson2D_Neumann.m for solving 2D Poisson's equation with homogeneous Neumann b.c. on a rectangle using the one-half grid.

(a) (20 points) In your report, describe your grid and numerical solution data structure. Write down the matrix vector form of your scheme. Find the eigendecomposition of the coefficient matrix.

(b) (10 points) Use $\Delta x = \Delta y$ in your code. Provide the loglog plot of the max norm error on 7 different grids for $Nx = 10, 20, \cdots, 70$ and compare it with the second order slope. Provide three contours for the exact solultion, numerical solution and absolute value of the pointwise error on the finest grid.

5. (bonus 20 points) Find the matrix $K3D$ by the kronecker product for solving $-u_{xx} - u_{yy} - u_{zz} = 14\pi^2 \sin(\pi x)\sin(2\pi y)\sin(3\pi z)$ with homogeneous Dirichlet b.c. on the domain $[0,1]\times[0,1]\times[0,1]$. The exact solution is $u(x,y,z) = \sin(\pi x)\sin(2\pi y)\sin(3\pi z)$. Implement the eigenvector method and test the convergence rate on several different grids. 3D array multiplications are not supported in MATLAB thus you have to reshape a 3D array to a 2D one in order to multiply a matrix to it. The following are examples of multiplying matrices Sx, Sy, Sz to each *dimension* of a 3D array F:

```

F=zeros(Nz,Ny,Nx);
F=reshape(F,Nz*Ny,Nx);
% Sx has size Nx by Nx; Sy has size Ny by Ny; Sz has size Nz by Nz
sol=kron(Sy,Sz)*F*Sx;

sol=reshape(sol, Nz,Ny,Nx);
F=reshape(F,Nz,Ny,Nx);
```