

The matrix-vector forms of the scheme is $\frac{1}{h^2}T_2U = \widehat{F}$ where \widehat{F} is the modified right hand side data due to the nonzero boundary conditions. Use eigenvector method to invert the matrix to receive the full credit. Use `inv(S)` in MATLAB to find S^{-1} . Multiply the eigenvector matrices S and S^{-1} directly. Do not consider the Discrete Cosine Transform.

- (a) (10 pts) Write out the scheme at the left boundary $x_1 = 0$ and find the local truncation error for the scheme at the left boundary.
- (b) (10 pts) Prove that stability of the scheme. Namely, let $A = \frac{1}{h^2}T_2$ then find $\|A^{-1}\|$ and show $\|A^{-1}\|$ is bounded by a constant (be rigorous and specific on what this constant is) as $h \rightarrow 0$. **Hint:** You can verify that T_2 is NOT a normal matrix thus you cannot use eigenvalues of T_2 to find $\|T_2^{-1}\|$. It is hard to find singular values of T_2 . On the other hand, we can find a diagonal matrix D s.t. $T_2 = DT$ where T matrix is symmetric (thus normal). Use eigenvalues of T and the inequality $\|DT\| \leq \|D\|\|T\|$ to prove the stability.
- (c) (10 pts) **In your report, give explicit expressions for \widehat{F} .** As in the sample code, provide the loglog plot of errors in max norm and 2-norm of the scheme with comparison to the second order slope line on 10 different grids: $n = 10, 20, 30, \dots, 100$. Also plot the numerical solutions V.S. the true solution on the finest grid $n = 100$ with the following range on the axis:

```

1          axis([0 1 -1 2.3])

```

3. (30 points) Design a second order finite difference scheme for solving

$$-u_{xx}(x, y) - u_{yy}(x, y) = \frac{5}{4}\pi^2 \sin\left(\frac{1}{2}\pi x\right) \cos(\pi y)$$

on the rectangle $(x, y) \in (0, 3) \times (0, 1.5)$ with the following boundary conditions:

- Dirichlet b.c. on the left and right boundaries: $u(0, y) = 0$, $u(3, y) = \sin\left(\frac{3}{2}\pi\right) \cos(\pi y)$.
- Neumann b.c. on the top and bottom boundaries: $\frac{\partial}{\partial y}u(x, 0) = 0$, $\frac{\partial}{\partial y}u(x, 1.5) = -\pi \sin\left(\frac{1}{2}\pi x\right) \sin(\pi y)$.

The exact solution is $u(x, y) = \sin\left(\frac{1}{2}\pi x\right) \cos(\pi y)$. Use the eigenvector method to implement the scheme. See `Poisson2D_Neumann.m` for solving 2D Poisson's equation with homogeneous Neumann b.c. on a rectangle using the one-half grid.

- (a) (20 points) In your report, describe your grid and numerical solution data structure. Write down the matrix vector form of your scheme. Find the eigendecomposition of the coefficient matrix.
- (b) (10 points) Use $\Delta x = \Delta y$ in your code. Provide the loglog plot of the max norm error on 7 different grids for $Nx = 10, 20, \dots, 70$ and compare it with the second order slope. Provide three contours for the exact solution, numerical solution and absolute value of the pointwise error on the finest grid.

4. (10 points+bonus 10 points) For solving $-u''(x) = f(x)$, $x \in (0, 1)$ with homogeneous Dirichlet boundary condition $u(0) = u(1) = 0$, first use a uniform interior grid $x_j = jh$ with spacing $h = \frac{1}{n+1}$ **where n must be odd**, then consider the following finite difference scheme:

$$\frac{-u_{j-1} + 2u_j - u_{j+1}}{h^2} = f_j, \quad \text{if } j \text{ is odd;}$$

$$\frac{u_{j-2} - 8u_{j-1} + 14u_j - 8u_{j+1} + u_{j+2}}{4h^2} = f_j, \quad \text{if } j \text{ is even.}$$

- (10 points) Find the truncation error of the scheme.
- (bonus 10 points) Implement this scheme for an exact solution $u(x) = \sin(\pi x) + 1.1\sin(2\pi x) - \sin(3\pi x)$ with $f(x) = -u''(x)$, and find its order of errors by plotting similar figures as in Problem 2(c) (you can plot a line with a proper slope). You can modify the sample code Poisson1D_Dirichlet.m provided on the course webpage if needed. For solving the linear system $AU = F$, you can simply use `mldivide` (*backslash*):

```
1 U= A \ F;
```

5. (bonus 20 points) Find the matrix K_{3D} by the kronecker product for solving $-u_{xx} - u_{yy} - u_{zz} = 14\pi^2 \sin(\pi x) \sin(2\pi y) \sin(3\pi z)$ with homogeneous Dirichlet b.c. on the domain $[0, 1] \times [0, 1] \times [0, 1]$. The exact solution is $u(x, y, z) = \sin(\pi x) \sin(2\pi y) \sin(3\pi z)$. Implement the eigenvector method and test the convergence rate on several different grids. 3D array multiplications are not supported in MATLAB thus you have to reshape a 3D array to a 2D one in order to multiply a matrix to it. The following are examples of multiplying matrices S_x, S_y, S_z to each *dimension* of a 3D array F :

```
1
2 F=zeros (Nz, Ny, Nx) ;
3 F=reshape (F, Nz*Ny, Nx) ;
4 % Sx has size Nx by Nx; Sy has size Ny by Ny; Sz has size Nz by Nz
5 sol=kron (Sy, Sz) *F*Sx;
6
7 sol=reshape (sol, Nz, Ny, Nx) ;
8 F=reshape (F, Nz, Ny, Nx) ;
```