Exam 1

Q1.

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ a & a & 3 & 0 \\ 0 & 1 & -2 & 2 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 - aR_1} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -a & 3+a & -a \\ 0 & 1 & -2 & 2 \end{array}\right)$$

$$\xrightarrow{R_2 \rightleftarrows R_3} \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & -a & 3+a & -a \end{pmatrix} \xrightarrow{R_3 = R_3 + aR_2} \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 2 \\ \underline{0} & \underline{0} & \underline{3-a} & \underline{a} \end{pmatrix}$$

$\left(\begin{array}{c|c} 0 \cdots\cdots 0 & b \end{array}\right)$, $b \neq 0 \Rightarrow$ no solution

$$\begin{cases} 3 - a = 0 & ① \\ a \neq 0 & ② \end{cases}, \quad ① \Rightarrow a = 3.$$

$A \sim B$, row equivalent

$A = \underbrace{E_1 E_2 \cdots E_n}_{\text{elementary matrices}} B$, $\longrightarrow$ apply a sequence of EROs.

① $A \sim B$, $\quad \text{col}(A) \overset{?}{=} \text{col}(B)$ ⊗

$\text{row}(A) \overset{?}{=} \text{row}(B)$ ✓

$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{col}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

$B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \text{col}(B) = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

② $A \sim B$, $\quad \text{ref}(A) = \text{ref}(B)$

$$\text{ref}(A) = \underbrace{F_1 F_2 \cdots F_n}_{\text{elementary matrices}} A$$

$$A = \underbrace{E_1 E_2 \cdots E_m}_{} B$$

$$\text{ref}(A) = F_1 F_2 \cdots F_n \cdot E_1 E_2 \cdots E_m B$$

$\Rightarrow \quad \text{ref}(A) = \text{ref}(B)$

$\Rightarrow$ (i) $\text{null}(A) = \text{null}(B)$

(ii) $\underline{\dim(\text{col}(A))} = \dim(\text{col}(B))$

# of leading cols of ref(A), but to construct the col(A), we need to A instead of ref(A).

Q2. $A \in \mathbb{R}^{m \times n}$.

(i) $Ax = b$ is consistent for all $b \in \mathbb{R}^m$,

$x_1 \vec{a_1} + x_2 \vec{a_2} + \dots + x_n \vec{a_n} = b$, (*)

$\vec{a_1}\ \vec{a_2} \dots \vec{a_n}$ are cols of A.

$\Rightarrow$ if $Ax = b$ is consistent

$\Rightarrow$ there exist $x_1 \dots x_n$ such that (*) is true.

$\Rightarrow$ $b \in \text{col}(A)$

for $b \in \mathbb{R}^m$, $b \in \text{col}(A)$ $\Rightarrow$ $\text{col}(A) \supseteq \mathbb{R}^m$

$\Rightarrow$ $\text{rank}(A) \geq m$ $\Rightarrow$ $m \leq \text{rank}(A) \leq n$

$n \geq \dim(\text{col}(A))$

(ii) <u>Rank theorem</u>

$$\underbrace{\text{rank}(A)}_{\dim(\text{col}(A))} + \underbrace{\text{nullity}(A)}_{\dim(\text{null}(A))} = \#\text{ of cols of } A.$$

$\text{null}(A) = \{0\}$

$\dim(\text{null}(A)) = 0$

by the Rank theorem $\Rightarrow$ $\text{rank}(A) = n$

iii $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}_{3\cdot 3}$

iv $\text{rank}(A) = \#\text{col} = \#\text{rows}$

$\Leftrightarrow$ $\det(A) \neq 0$ $\Leftrightarrow$ $Ax = 0$, has only

trivial solution $\Leftrightarrow$ $Ax = b$ has unique solution

$\Leftrightarrow$ the cols of A are linearly indep.

V. T is 1-1 linear transformation

$\Leftrightarrow$ cols of A are linearly indep.

$\Rightarrow$ cols of A are linearly indep

$\Rightarrow$ rank (A) = # cols = n.

$\begin{array}{c} \text{rank}(A) \\ \| \ \text{given} \\ m \end{array}$

$\Rightarrow$ $m = n$ $\Rightarrow$ square matrix with rank = n.

3. B:

T is linear, T is 1-1 iff $T(x) = 0$
$\Leftrightarrow$
has only trivial solution. $\Leftrightarrow$ $\ker(T) = \{0\}$

$\text{null}(A) = \{x, \ Ax = 0\}$
$\ker(T) = \{x, \ T(x) = 0\}$
generalization of null (A)

A. Assume $T(v_1)$ & $T(v_2)$ are linearly dep.

def $\Rightarrow$ there exist c_1 & c_2 not all $= 0$

such that $c_1 T(v_1) + c_2 T(v_2) = 0$

linear transformation

$T(a\vec{v} + b\vec{u})$

$= aT(\vec{v}) + bT(\vec{u})$

$\vec{v}$ $\vec{u}$ are in the domain

& $a, b \in \mathbb{R}$.

$(($ $T(c_1 v_1 + c_2 v_2) = 0$

b/c T is 1-1,

$\Rightarrow$ $c_1 v_1 + c_2 v_2 = 0$

def $\Rightarrow$ v_1 & v_2 are linear dep., contradict$^{\underline{n}}$.

C. $T: \mathbb{R}^n \to \mathbb{R}^m$, $A \in \mathbb{R}^{m \cdot n}$

$$T(x) = \underset{m \cdot n}{A} \cdot x \in \mathbb{R}^n$$

1-1 $\Leftrightarrow$ col(A) are linearly indep.

$\Rightarrow$ rank(A) $= n$.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

cols > # rows

row $\geq$ # cols

E. for all $b \in \mathbb{R}^m$, $Ax = b$ is consistant

$\Rightarrow \quad col(A) = \mathbb{R}^m$

$\underbrace{n}_{\#\text{ of col s}} = \underbrace{\dim(col(A))}_{} = m$ contradicts with $\underbrace{m > n}_{(C)}$

Q6. $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ $\det(A) = -2$

$$B = \begin{pmatrix} 3a-5b+c & 3d-5e+f & 3g-5h+i \\ 2c & 2f & 2i \\ b & e & h \end{pmatrix}$$

$\det(\downarrow) = -2$ $\qquad \det(\quad) = -2 \cdot 3$

$$A \rightarrow \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} \rightarrow \begin{pmatrix} \boxed{3}a & \boxed{3}d & \boxed{3}g \\ b & e & h \\ c & f & i \end{pmatrix}$$

$\det(\downarrow) = -6$ $\qquad \det(\downarrow) = 6$

$$\rightarrow \begin{pmatrix} 3a+c & 3d+f & 3g+i \\ \cdots & & \\ \cdots & & \end{pmatrix} \rightarrow \begin{pmatrix} 3a+c & 3d+f & 3g+i \\ c & f & i \\ b & e & h \end{pmatrix}$$

$\det(\downarrow) = 2 \cdot 6 = 12$ $\qquad \det(B) = 12$.

$$\rightarrow \begin{pmatrix} \cdots & & \\ 2c & 2f & 2i \\ \cdots & & \end{pmatrix} \rightarrow B$$

$$\det(2A) = 2^{\#\text{ of rows}} \det(A)$$

6.5 HW 1.

$$A = \begin{pmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{pmatrix} \qquad b = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$$

compute the least square sol of $Ax = b$.

$$A^t A = \begin{pmatrix} 6 & -11 \\ -11 & 22 \end{pmatrix}$$

$$A^t b = \begin{pmatrix} -4 \\ 11 \end{pmatrix}$$

$$(C \mid I) \xrightarrow{\text{ERO}} (I \mid C^{-1})$$

normal equation

$$(A^t A) \cdot x = A^t b$$

If $A^t A$ is invertible

$$x = (A^t A)^{-1} \cdot A^t b$$

$$\boxed{C = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow C^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

$$\hat{x} = \frac{1}{6 \cdot 22 - 11 \cdot 11} \begin{pmatrix} 22 & 11 \\ 11 & 6 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 11 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$