MA 265, Spring 2022, Midterm 2 (GREEN)

INSTRUCTIONS:

1. **Write your answers of the seven multiple choice questions into the table on the last page. Show all your work on the questions and you may use the back of the test pages as scratch paper if needed.**
2. After you have finished the exam, hand in your test booklet to your instructor.

172	MWF	11:30AM	Zhang, Ying
173	MWF	12:30PM	Zhang, Ying
185	TR	12:00PM	Tsymbaliuk, Oleksandr
196	TR	1:30PM	Tsymbaliuk, Oleksandr
201	MWF	11:30AM	Debray, Arun
202	TR	12:00PM	Zhang, Zecheng
213	TR	4:30PM	Zhang, Zecheng
214	MWF	4:30PM	Xu, Xuefeng
225	MWF	3:30PM	Xu, Xuefeng
226	MWF	10:30AM	Yhee, Farrah
237	MWF	11:30AM	Yhee, Farrah
238	TR	9:00AM	Yang, Guang
240	TR	10:30AM	Yang, Guang
241	TR	3:00PM	Noack, Christian james
252	TR	4:30PM	Noack, Christian james
253	TR	4:30PM	Kadattur, Shuddhodan
264	TR	3:00PM	Kadattur, Shuddhodan
265	MWF	3:30PM	Nguyen, Thi-Phong
276	MWF	4:30PM	Nguyen, Thi-Phong
277	TR	12:00PM	Zhang, Qing
281	TR	1:30PM	Zhang, Qing
282	MWF	1:30PM	Tang, Shiang
283	MWF	1:30PM	Zhang, Ying
284	MWF	2:30PM	Debray, Arun
285	MWF	2:30PM	Tang, Shiang
287	TR	9:00AM	Rivera, Manuel
288	MWF	10:30AM	Mohammad-Nezhad, Ali
289	MWF	11:30AM	Mohammad-Nezhad, Ali
290	TR	10:30AM	Miller, Jeremy
291	TR	12:00PM	Ulrich, Bernd
292	MWF	3:30PM	Heinzer, William

3. NO CALCULATORS, BOOKS, NOTES, PHONES OR CAMERAS ARE ALLOWED on this exam. Turn off or put away all electronic devices.
4. **Please remain seated during the last 10 minutes of the exam.** When time is called, all students must put down their writing instruments immediately. You may remain in your seat while your instructor will collect the exam booklets.
5. Anyone who violates these instructions will have committed an act of academic dishonesty. Penalties for such behavior can be severe and may include an automatic F on the course. All cases of academic dishonesty will be reported to the Office of the Dean of Students.

I have read and understand the above instructions regarding academic dishonesty:

STUDENT NAME ____________________

STUDENT SIGNATURE ____________________

STUDENT PUID ____________________

SECTION NUMBER ____________________

1. (10 points) Which of the following statements is not always true?

A. If u, v, and w are linearly independent vectors in a vector space, then $u+v$, $v+w$, and w are linearly independent.

B. Every linearly independent set of $\mathbb{R}^n$ consists of at most n vectors.

C. Every spanning set of $\mathbb{R}^n$ contains a basis of $\mathbb{R}^n$.

D. If the nullity of a matrix A is zero, then linear system $Ax = b$ has a unique solution for every b.

E. Any integer between 1 and 4 can be the rank of a 6×4 matrix.

Correct Answer is: **D**

2. (10 points) Let $\mathbb{M}_{3\times 3}$ be the vector space of all 3×3 matrices, and let H be its subspace consisting of all A satisfying

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} A = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

What is the dimension of H?

A. 1

B. 2

C. 3

D. 4

E. 5

Correct Answer is: **C**

3. (10 points) For which number(s) a does the matrix $A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ -2 & -1 & 0 & 0 \\ 10 & -9 & 6 & a \\ 1 & 5 & a & 3 \end{bmatrix}$ have 2 as an eigenvalue?

A. $a = 3$ only

B. $a = 3$ and $a = -3$ only

C. $a = 2$ only

D. $a = 2$ and $a = 3$ only

E. $a = 2$ and $a = -2$ only

Correct Answer is: **E**

how to show/check if a matrix $A \in \mathbb{R}^{n \times n}$ is diagonalizable?

① def: there exists an invertible P & a diag matrix D, such that $A = PDP^{-1}$.

{ $D \to$ eigenvalues on the diagonal
$P \to$ corresponding e.vectors

You need to find n linearly indep e-vectors.

② symmetric or not.
↳ $\Rightarrow$ orthogonally diagonalizable

③ for triangular matrix, if diagonal entries are distinct, $\Rightarrow$ matrix is diagonalizable.

4. (10 points) Let A and B be similar $n \times n$ matrices with real entries. Which of the following statements must be TRUE?

$A = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 6 & 3 \end{pmatrix}$ $\lambda_1 = 1$ ($\vec{v}_1$) $\lambda_2 = 2$ ($\vec{v}_2$) $\lambda_3 = 3$ ($\vec{v}_3$)

thm: the eigen-vectors of the different eigenvalues are linearly independent.

(i) A and B have the same characteristic polynomial.

(ii) If the columns of A are linearly independent, then 0 is an eigenvalue of A.

(iii) If A is diagonalizable, then all the eigenvalues of A must be nonzero.

(iv) If $-\lambda$ is an eigenvalue of A, then λ^4 is an eigenvalue of B^4.

(v) If A is diagonalizable, then B is diagonalizable.

$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

{ $\lambda_1 = 1$
$\lambda_2 = 2$
$\lambda_3 = 0$

A is diagonalizable.

$\{\vec{v}_1\ \vec{v}_2\ \vec{v}_3\}$ are linearly indep. $\Rightarrow P = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$

A. (i), (ii), (iv), (v)

B. (i), (iii), (iv), (v)

C. (iv), (v)

D. (i), (iv), (v)

E. All of the statements are true.

Correct Answer is: **D**

④ a square matrix A is diagonalizable iff A^t is diagonalizable.

$(CD)^t = D^t C^t$

$A = PDP^{-1}$

$A^t = (PDP^{-1})^t = (P^{-1})^t \cdot (PD)^t$

$(P^{-1})^t = (P^t)^{-1}$

$= \underbrace{(P^t)^{-1}}_{Q} \cdot D^t \cdot P^t$

$= Q^{-1} D^t Q$

↓ ↓ ↓

⑤ $A \in \mathbb{R}^{m \times n}$, $B = A^t A$, $\Rightarrow B$ is diagonalizable.

pf: $B^t = (A^t A)^t$

$= A^t (A^t)^t = A^t A$

$= B$

$\Rightarrow B$ is symmetric.

5. (10 points) Consider the differential equation

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

Then the origin is

A. an attractor

B. a repeller

C. a saddle point

D. a spiral point

E. none of the above

Correct Answer is : **B**

6. (10 points) Let $\mathbb{P}_3$ denote the vector space of all polynomials of degree at most 3, which of the following subsets are subspaces of either $\mathbb{R}^3$ or $\mathbb{P}_3$?

(i) The set of all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$ such that $x + 2y + 3z = 1$.

(ii) The set of all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$ such that $10x - 2y = z$.

(iii) The set of all polynomials $p(t)$ in $\mathbb{P}_3$ such that the degree of $p(t)$ is 3.

(iv) The set of all polynomials $p(t)$ in $\mathbb{P}_3$ satisfying $p(2) = 0$.

A. (ii) and (iv) only

B. (i), (ii) and (iv) only

C. (ii) and (iii) only

D. (ii), (iii) and (iv) only

E. (iii) and (iv) only

Correct Answer is : **A**

7. (10 points) A real 2×2 matrix A has an eigenvalue $\lambda_1 = 1 + i$ with corresponding eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 - 2i \\ 3 + 4i \end{bmatrix}$. Which of the following is the general **REAL** solution to the system of differential equations $\mathbf{x}'(t) = A\mathbf{x}(t)$?

A. $c_1 e^t \begin{bmatrix} \cos t + 2\sin t \\ 3\cos t - 4\sin t \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin t - 2\cos t \\ 3\sin t + 4\cos t \end{bmatrix}$

B. $c_1 e^t \begin{bmatrix} \cos t - 2\sin t \\ 3\cos t + 4\sin t \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin t + 2\cos t \\ 3\sin t - 4\cos t \end{bmatrix}$

C. $c_1 e^t \begin{bmatrix} \cos t + 2\sin t \\ 3\cos t + 4\sin t \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin t - 2\cos t \\ 3\sin t - 4\cos t \end{bmatrix}$

D. $c_1 e^t \begin{bmatrix} -\cos t + 2\sin t \\ -3\cos t - 4\sin t \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin t - 2\cos t \\ 3\sin t + 4\cos t \end{bmatrix}$

E. $c_1 e^t \begin{bmatrix} \cos t + 2\sin t \\ 3\cos t - 4\sin t \end{bmatrix} + c_2 e^t \begin{bmatrix} -\sin t - 2\cos t \\ -3\sin t + 4\cos t \end{bmatrix}$

Correct Answer is : **A**

8. Let $\mathbb{P}_2$ denote the vector space of all polynomials of degree at most 2 in the variable t, and let $\mathbb{M}_{2\times 2}$ denote the vector space of all 2×2 matrices. Consider a linear transformation:

$$T\colon \mathbb{P}_2 \to \mathbb{M}_{2\times 2} \quad \text{given by} \quad T(p(t)) = \begin{bmatrix} p(0) & p'(0) \\ p(1) & p'(1) \end{bmatrix}.$$

(1) (3 points) Find $T(at^2+bt+c)$.

$$T(at^2+bt+c) = \begin{bmatrix} c & b \\ a+b+c & 2a+b \end{bmatrix}.$$

(2) (3 points) Find a polynomial $p(t)$ in $\mathbb{P}_2$ such that $T(p(t)) = \begin{bmatrix} 1 & 2 \\ 4 & 4 \end{bmatrix}$.

$p(t) = t^2 + 2t + 1.$

(3) (4 points) Find a basis for the range of T.

$\left\{ \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ *is a basis for the range of T. (Answer may vary!)*

9. (6 points) (1) Find all the eigenvalues of matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 5 & 1 \\ -1 & -3 & 1 \end{bmatrix}$, and find a basis for the eigenspace corresponding to each of the eigenvalues.

$\lambda_1 = 4$, basis for the eigenspace $\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

$\lambda_2 = \lambda_3 = 2$, basis for the eigenspace $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ (Answer may vary)

(4 points) (2) Find an invertible matrix P and a diagonal matrix D such that

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 5 & 1 \\ -1 & -3 & 1 \end{bmatrix} = PDP^{-1}.$$

$P = \begin{bmatrix} 0 & -3 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $\quad D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (Answer may vary)

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 5 & 1 \\ -1 & -3 & 1 \end{pmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\det \begin{pmatrix} 2-\lambda & 0 & 0 \\ 1 & 5-\lambda & 1 \\ -1 & -3 & 1-\lambda \end{pmatrix} = 0$$

$$(2-\lambda) \det \begin{pmatrix} 5-\lambda & 1 \\ -3 & 1-\lambda \end{pmatrix} = 0$$

$$(2-\lambda)\left((\lambda-1)(\lambda-5) + 3 \right) = 0$$

$$(\lambda-2)\left(\lambda^2 - 5\lambda - \lambda + 5 + 3 \right) = 0$$

$$(\lambda-2)(\lambda^2 - 6\lambda + 8) = 0$$

$$(\lambda-2)(\lambda-2)(\lambda-4) = 0$$

$$\begin{cases} \lambda_1 = \lambda_2 = 2 \\ \lambda_3 = 4 \end{cases}$$

$\lambda_1 = \lambda_2 = 2$

Find $v \neq 0$, s.t.

$$(A - \lambda I) v = 0$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 3 & 1 \\ -1 & -3 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\downarrow$$

$$\begin{array}{c} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & -3 & -1 \end{pmatrix} \\ \begin{array}{ccc} & \uparrow & \uparrow \\ & s & t \end{array} \end{array} \Rightarrow -x_1 - 3x_2 - x_3 = 0$$

$$x_1 = -3s - t$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3s - t \\ s \\ t \end{pmatrix} = \begin{pmatrix} -3s \\ s \\ 0 \end{pmatrix} + \begin{pmatrix} -t \\ 0 \\ t \end{pmatrix}$$

$$= +s \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$v_1 = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$ $v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ are 2 linearly indep eigen-vectors. //

$\lambda_3 = 4.$ $v_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

$$P = \begin{pmatrix} -3 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

//

Trace of a matrix (square)

def $\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}$, $A \in \mathbb{R}^{n \cdot n}$.

$a_{ij} \to$ i th row j th col entry

$\rightarrow$ summation of all diagonal entries.

1

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$tr(A) = a + d$$

$$det(A) = ad - bc$$

eigenvalues

$$det(A - \lambda I) = 0$$

$$det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0$$

$$(\lambda - a)(\lambda - d) - bc = 0$$

$$\lambda^2 - d\lambda - a\lambda + ad - bc = 0$$

$$\lambda^2 - (d + a)\lambda + (ad - bc) = 0$$

$$\lambda^2 - tr(A)\lambda + det(A) = 0.$$

(2) $\operatorname{tr}(A) = \operatorname{tr}(A^t)$

(3) $\operatorname{tr}(AB) = \operatorname{tr}(BA)$

(4) suppose A is diagonalizable.

$$A = PDP^{-1}$$

$$\operatorname{tr}(A) = \operatorname{tr}(\underbrace{PD}_{C}\,\underbrace{P^{-1}}_{D})$$

$$\overset{(3)}{=} \operatorname{tr}(P^{-1} \cdot PD) = \operatorname{tr}(D)$$

= summation of all e.values.

If we further assume A is invertable.

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \left| \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \right)$$

$$A = PDP^{-1}$$

$$A^{-1} = (P^{-1})^{-1} \cdot (PD)^{-1} = PD^{-1}P^{-1}$$

$$D^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}$$

$$\operatorname{tr}(A^{-1}) = \operatorname{tr}(D^{-1})$$

linear transformation & A.

$T: \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation.

there exists a unique matrix $A \in \mathbb{R}^{n \cdot m}$

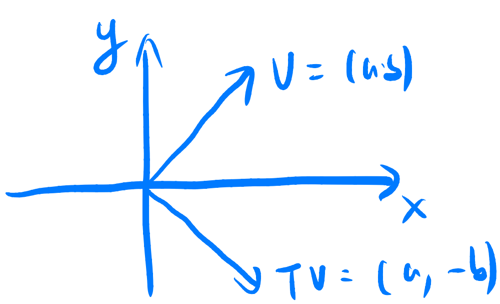
s.t. $T(x) = \underbrace{Ax}_{[A]_{n\cdot m}(x)_{m\cdot 1}}$

$$A = [\, T(\vec{e}_1) \cdots \quad T(\vec{e}_m) \,]$$

$\vec{e}_1, \ldots \vec{e}_m$ are std basis of $\mathbb{R}^m$

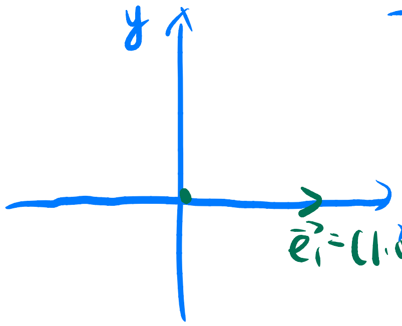
$$\vec{e}_j = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow \text{j th entry.}$$

$T: \mathbb{R}^2 \to \mathbb{R}^2$, reflection through x-axis

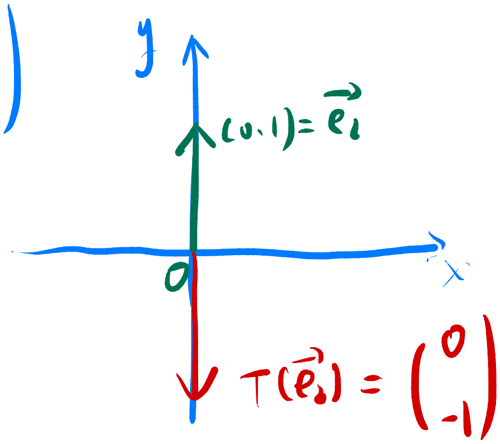


Find A matrix.

Std basis $\vec{e}_1 = \begin{pmatrix}1\\0\end{pmatrix}$ $\vec{e}_2 = \begin{pmatrix}0\\1\end{pmatrix}$

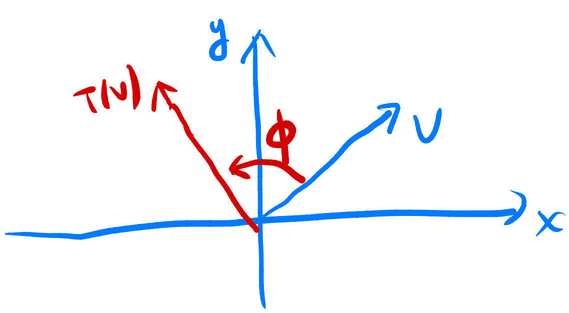


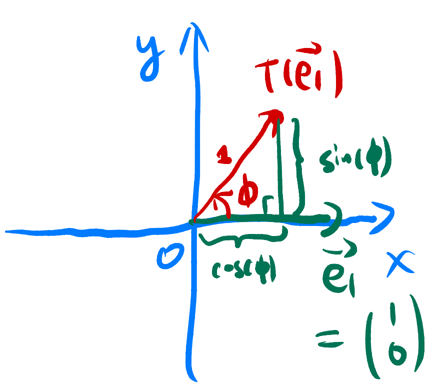
$T(\vec{e}_1) = \begin{pmatrix}1\\0\end{pmatrix}$



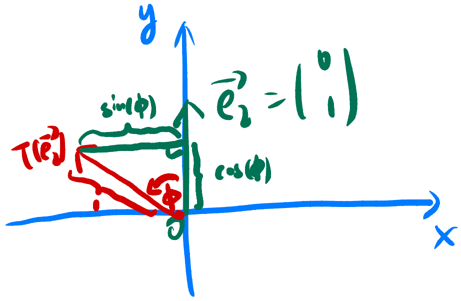
$$A = \begin{pmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

T: Rotation counter-clockwise through angle ϕ.





$T(\vec{e}_1) = \begin{pmatrix}\cos(\phi)\\ \sin(\phi)\end{pmatrix}$



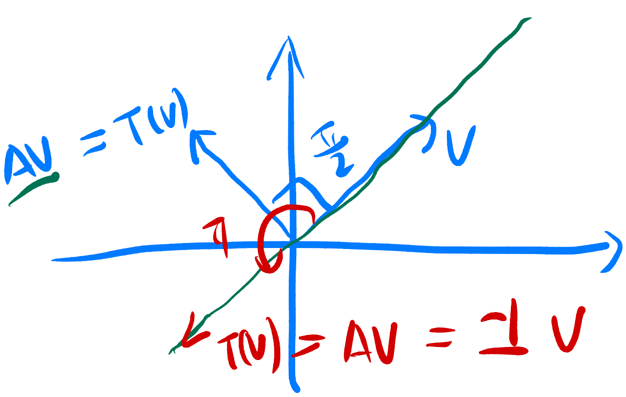
$$T(\vec{e}_2) = \begin{pmatrix} -\sin(\phi) \\ \cos(\phi) \end{pmatrix}$$

$$A = \begin{pmatrix} \cos(\phi), & -\sin(\phi) \\ \sin(\phi), & \cos(\phi) \end{pmatrix}$$

$$\underset{\substack{\downarrow \\ T(v)}}{Av} = \underset{\substack{\downarrow \\ \text{scaling}}}{\lambda v} \longleftrightarrow \text{eigen-problem.}$$

Rotate counterclockwise through an angle $\frac{\pi}{2}$.

$$\longrightarrow \quad A = \begin{pmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{pmatrix}$$



$\Rightarrow$ there will be no real eig-value.

$\lambda = -1$

rotate .... angle $= \pi$

10. (4 points) (1) Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 9 & 5 \\ -6 & -2 \end{bmatrix}.$$

Correct Answer is : $\lambda_1 = 3, \mathbf{v}_1 = \begin{bmatrix} 5 \\ -6 \end{bmatrix}, \lambda_2 = 4, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ *(Answer may vary)*

(2 points) (2) Find a general solution to the system of differential equations

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ -6 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

Correct Answer is : $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{3t} \begin{bmatrix} 5 \\ -6 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ *(Answer may vary)*

(4 points) (3) Let $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ be a particular solution to the initial value problem

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ -6 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \qquad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Find $x(1) + y(1)$.

Correct Answer is : $c_1 = -1, c_2 = 6, x(1) + y(1) = e^3$

Please write your answers of the 7 multiple choice questions in the following table.

Question	Answer
1. (10 points)	
2. (10 points)	
3. (10 points)	
4. (10 points)	
5. (10 points)	
6. (10 points)	
7. (10 points)	

Total Points: ______________________