Section 1.1 Systems of Linear Equations

Definitions

(unknown)

1. The equation in the variables $x_1, \cdots, x_n$ that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = \underline{b} \quad \text{constant} \tag{1}$$

is <u>a linear equation.</u>, $a_1, \cdots, a_n$ are called <u>coefficients</u>. b and a_i-s are known real or complex numbers.

2. A collection of one or more linear equations involving the same variables $x_1, \cdots, x_n$ are called a system of linear equation , For example,

linear system

$$\begin{cases} x_1 + 2x_2 + 0\cdot x_3 = 0 \\ 0\cdot x_1 + x_2 + 4x_3 = 0 \\ x_1 + x_2 + x_3 = 1 \end{cases}$$

3. A sequence of numbers $s_1, s_2, \cdots, s_n$ such that (1) is satisfied when $x_1 = s_1, x_2 = s_2, \cdots, x_n = s_n$ is called a solution of the system.

4. The set of all possible solutions is called the solution set of the linear system.

5. Two linear systems are equivalent if they have the same solution set.

6. A system of linear equation has
   (a) no solution, $\longrightarrow$ inconsistent.
   (b) exactly one solution, or
   (c) infinitely many solutions. $\Big\}$ consistent.

7. A system of linear equation is consistent if it has either one solution or infinitely many solutions. A system is inconsistent if it has no solution.

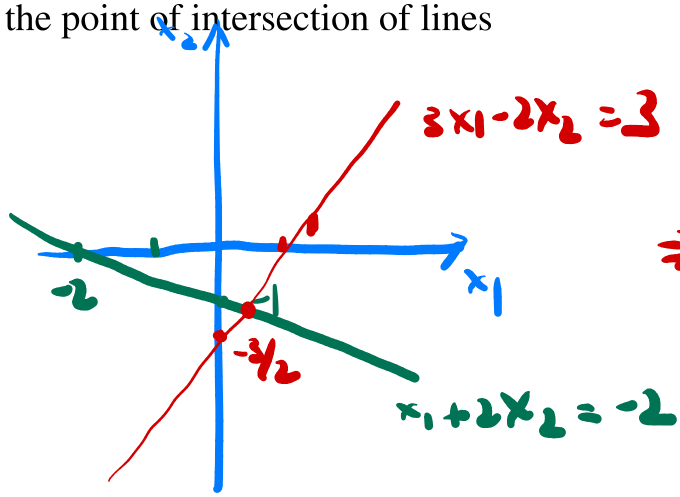
8. Finding the solution set of a system of two linear equations in two variables. $\iff$ Finding the intersections of two lines.

Example 1: Find the point of intersection of lines

(a)

1 $x_1 + 2x_2 = -2$

2 $3x_1 - 2x_2 = 3$

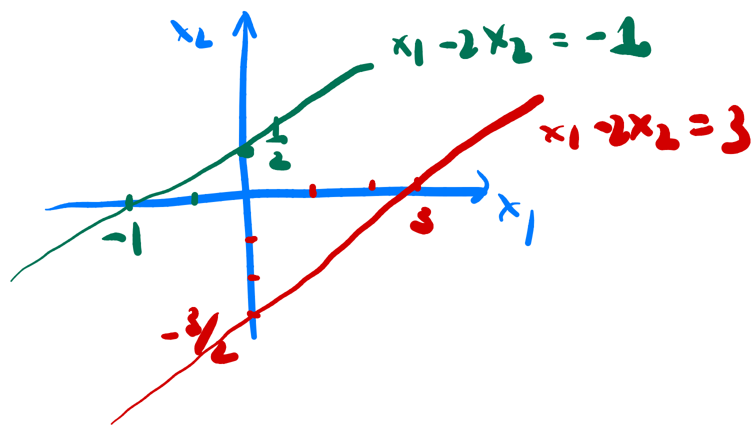


⇒ the linear system has unique solution.

(b)

1 $x_1 - 2x_2 = -1$

2 $x_1 - 2x_2 = 3$



parallel ⇒ no solution

(c)

$$
\begin{aligned}
x_1 - 2x_2 &= -1 \\
-x_1 + 2x_2 &= 1
\end{aligned}
$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}_{2\cdot 3}$$

3 columns

2 rows

Matrix Notation

The essential information of a linear system can be recorded compactly in a rectangular array called a matrix. The size of a matrix tells how many rows and columns it has.

Example 2:

$$\begin{aligned} x_1 + 4x_2 - 2x_3 + 8x_4 &= 12 \\ 0\cdot x_1 + x_2 - 7x_3 + 2x_4 &= -4 \\ 5x_3 - x_4 &= 7 \\ x_3 + 3x_4 &= -5 \end{aligned} \tag{2}$$

The **coefficient matrix** of (2) is

coeff. of x_1 x_2 x_3 x_4 of the 1st row

$$\begin{pmatrix} 1 & 4 & -2 & 8 \\ 0 & 1 & -7 & 2 \\ 0 & 0 & 5 & -1 \\ 0 & 0 & 1 & 3 \end{pmatrix}_{4\cdot 4}$$

# of rows = # of eqns

# of cols = # of variables

The **augmented matrix** of (2) is

$$\left(\begin{array}{cccc:c} 1 & 4 & -2 & 8 & 12 \\ 0 & 1 & -7 & 2 & -4 \\ 0 & 0 & 5 & -1 & 7 \\ 0 & 0 & 1 & 3 & 5 \end{array}\right)_{4\cdot 5}$$

Solving a linear system

Basic strategy: replace one system with an equivalent system that is easier to solve.
Example 3: Solve system

$$\begin{aligned} x_1 - 2x_2 &= 0 \\ 2x_1 - 8x_2 &= 8 \end{aligned} \tag{3}$$

Elementary row operations (EROs)

1 (Replacement) Replace one row by the sum of itself and a multiple of another row

2 (Interchange) Interchange two rows

3 (Scaling) Multiply all entries in a row by a nonzero constant

Remarks:

1 Two matrices are called row equivalent if there is a sequence of elementary row operations that transforms one matrix into the other.

2 If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Existence and Uniqueness

Two Fundamental questions about a linear system:

1 Is the system consistent; that is, does at least one solution exist?

2 If a solution exists, is the solution unique?

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \xrightarrow[R_2 = R_2 - 4 \cdot R_1]{\text{replacement}} \begin{pmatrix} 1 & 2 & 3 \\ 4-4\cdot 1, & 5-4\cdot 2, & 6-4\cdot 3 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$\xrightarrow[R_3 = R_3 - 7R_1]{} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 8-7\cdot 2 & 9-7\cdot 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix}$$

$$\xrightarrow[R_3 = R_3 - 2 \cdot R_2]{} \underset{\text{(REF)}}{\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix}} \xrightarrow[R_2 = R_2 \cdot (-\frac{1}{3})]{\text{scaling}} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Example 4: The augmented matrix of a linear system has been reduced by row operations to the form shown. Continue the appropriate row operations and describe the solution set of the original system.

$$\begin{array}{c}
\begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\
\left[\begin{array}{ccc|c}
1 & 7 & 3 & -4 \\
0 & 1 & -1 & 3 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & -2
\end{array}\right]
\end{array}$$

constants of each equation

$$\begin{cases}
x_1 + 7x_2 + 3x_3 = -4 \\
x_2 - x_3 = 3 \\
\underbrace{0x_1 + 0x_2 + 0x_3}_{=0} = 1 \\
x_3 = -2
\end{cases}$$

$= 1 \Rightarrow$ the system has no solution.