# Section 1.2 Row Reduction And Echelon Forms

## Definitions

1 is the leading entry of the 1st row

$$\begin{pmatrix} \boxed{1} & 2 & 3 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{pmatrix}$$

→ non-zero rows

→ leading entry.

1. A nonzero row or column in a matrix means a row or column that contains at least one nonzero entry; a leading entry of a row refers to the leftmost nonzero entry (in a nonzero row).

2. A rectangular matrix is in ___row___ echelon form (REF) or row echelon form if it has the following three properties:

   - All nonzero rows are above any rows of all zeros. (zero rows are at the bottom of the matrix)
   - Each leading entry of a row is in a column to the right of the leading entry of the row above it.
   - All entries in a column below a leading entry are zeros. (Property 3 is a simple consequence of property 2, but we include it for emphasis.)

3. If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced** row **echelon form** (or reduced row echelon form, we usually call it rref): (RREF)

   3.1 • The leading entry in each nonzero row is 1.
   3.2 • Each leading 1 is the only nonzero entry in its column.

Example 1:

$\blacksquare \to \neq 0$

$* \to$ any number

REF

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

RREF

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

**Theorem**: Uniqueness of the Reduced Echelon Form

Each matrix is row equivalent to one and only one reduced echelon matrix.

**Definitions**

**pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A.

A **pivot column** is a column of A that contains a pivot position.

**The Row Reduction Algorithm** ( algo will give us REF or RREF of the original matrix)

STEP 1: Begin with the leftmost nonzero column. This is a **pivot column**. The **pivot position** is at the top.

STEP 2: Select a nonzero entry in the pivot column as a **pivot**. If necessary, interchange rows to move this entry into the pivot position.

STEP 3: Use **row replacement operations** to create zeros in all positions below the pivot.

STEP 4: Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1 to 3 to the submatrix that remains. **Repeat** the process until there are no more nonzero rows to modify.

STEP 5: **Backward phase**. Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

Example 2: Row reduce the matrix $A$ below to echelon form, and locate the pivot columns of $A$.

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 4 & 8 \end{pmatrix} \xrightarrow[R_1 \& R_2]{\text{interchange}} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 4 & 8 \end{pmatrix} \xrightarrow[R_3 = R_3 - 3R_1]{\text{replacement}} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & \boxed{-2} & -1 \end{pmatrix}$$

leading entries

$$\xrightarrow[R_3 = R_3 + 2 \cdot R_2]{\text{replacement}} \begin{pmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & \boxed{3} \end{pmatrix}$$ ✓ REF

3.1 is satisfied.

Go to RREF

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix} \xrightarrow[R_3 = \frac{1}{3} \cdot R_3]{} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & \boxed{2} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[R_2 = R_2 - 2 \cdot R_3]{\text{make 2 in the 2nd row 3rd col} = 0} \begin{pmatrix} 1 & 2 & \boxed{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_1 = R_1 - 3 \cdot R_3]{\text{make 3 in the 1st row 3rd col} = 0}$$

$$\begin{pmatrix} 1 & \boxed{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_1 = R_1 - 2 \cdot R_2]{} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ (RREF)}$$

$$\left(\begin{array}{ccc|c} 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 \\ 0 & 2 & 3 & 4 \end{array}\right)_{3\cdot 4}$$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ x_1 & x_2 & x_3 \end{array}$$

3 eqns = # of rows

$$\xrightarrow{R_2 = R_2 - 3\cdot R_1} \begin{pmatrix} 2 & 3 & 4 & 5 \\ 0 & -2 & -4 & -6 \\ 0 & 2 & 3 & 4 \end{pmatrix} \xrightarrow{R_3 = R_3 + R_2} \begin{pmatrix} 2 & 3 & 4 & 5 \\ 0 & -2 & -4 & -6 \\ 0 & 0 & -1 & -2 \end{pmatrix}$$

$6 \to 0$

$2 \to 0$

(REF)

$$\xrightarrow[R_3 = R_3\cdot(-1)]{R_2 = R_2 \cdot -\frac{1}{2}} \begin{pmatrix} 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix} \xrightarrow{R_2 = R_2 - 2R_3} \begin{pmatrix} 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$2 \to 0$

$$\xrightarrow{R_1 = R_1 - 4R_3} \begin{pmatrix} 2 & 3 & 0 & -3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \xrightarrow{R_1 = R_1 - 3R_2} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$4 \to 0$

$3 \to 0$

$$\xrightarrow{R_1 = \frac{1}{2}R_1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$2 \to 1$

RREF.

$$\begin{pmatrix} 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 \\ 0 & 2 & 3 & 4 \end{pmatrix} \xrightarrow[6 \to 0]{R_2 = R_2 - 3R_1} \begin{pmatrix} 2 & 3 & 4 & 5 \\ 0 & -2 & -4 & -6 \\ 0 & 2 & 3 & 4 \end{pmatrix}$$

$$\xrightarrow[2 \to 0]{R_3 = R_3 + R_2} \begin{pmatrix} 2 & 3 & 4 & 5 \\ 0 & -2 & -4 & -6 \\ 0 & 0 & -1 & -2 \end{pmatrix} \quad \text{(REF)}$$

$$\xrightarrow[R_3 = -1 \cdot R_3]{R_2 = -\frac{1}{2} R_2} \begin{pmatrix} 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\xrightarrow[2 \to 0]{R_2 = R_2 - 2R_3} \begin{pmatrix} 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\xrightarrow[4 \to 0]{R_1 = R_1 - 4R_3} \begin{pmatrix} 2 & 3 & 0 & -3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \xrightarrow[3 \to 0]{R_1 = R_1 - 3R_2}$$

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \xrightarrow[2 \to 1]{R_1 = \frac{1}{2} R_1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

## Solutions of Linear Systems

Definition:

Basic variable: the variables correponding to pivot columns in the matrix (leading entry $\leftrightarrow$ basic variable)

Free variable: the other variables $\Rightarrow$ acts as the parameter, determines the solution of the system.

Example 3: The augmented matrix of a linear system has been written in the reduced echelon form as follows

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(1) Determine the basic variable and free variable. $\rightarrow$ 3 variables, 3 equations

leading entries $\Rightarrow \begin{cases} x_1 \\ x_2 \end{cases}$ basic variables

$x_3 \rightarrow$ free variable.

(2)Find the general solution of the above system.

set $x_3 = s$ (free variable), $s$ can be any number.

from the $2^{nd}$ row $\quad x_2 + x_3 = 4$

$\Rightarrow \quad x_2 = 4 - x_3 = 4 - s$

from the $1^{st}$ row $\Rightarrow \quad x_1 - 5x_3 = 1$

$x_1 = 1 + 5x_3 = 1 + 5s$

parametric form of the solution

$$\begin{cases} x_1 = 1 + 5s \\ x_2 = 4 - s \\ x_3 = s \end{cases}$$
, $s$ is the parameter & can be any number.

**Remark 1** *We have some remarks:*

1. *Each different choice of $x_3$ determines a (different) solution of the system, and every solution of the system is determined by a choice of $x_3$.*

2. *Parametric descriptions of solution sets. The free variables act as parameters. Solving a system amounts to finding a parametric description of the solution set or determining that the solution set is empty.*

3. *Whenever a system is consistent and has free variables, the solution set has many parametric descriptions. Whenever a system is inconsistent, the solution set is empty, even when the system has free variables. In this case, the solution set has no parametric representation.*

**Theorem: Existence and Uniqueness Theorem**

A linear system is consistent consistent if and only if the rightmost column of the augmented matrix is not a pivot column —that is, if and only if an echelon form of the augmented matrix has no row of the form

$$[0 \quad \cdots \quad 0 \quad b]$$

constant in the equation

$b \neq 0$

$[0 \quad 0 \quad 0 \quad 1]$

$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 1$

$\Rightarrow$ no solution

$\Rightarrow$ inconsistent.

with $b$ nonzero.

If a linear system is consistent, then the solution set contains either

- unique solution, when there is no free variable, or,
- infinitely many solutions, when there is at least one $\underbrace{\text{free variable}}$.

free variable determines the solution, it's parameter

Using Row Reduction To Solve A Linear System

Step 1: Write the **augmented matrix** of the system.

Step 2: Use the **row reduction algorithm** to obtain an equivalent augmented matrix in **echelon form**. Decide whether the system is **consistent**. If there is no solution, stop; otherwise, go to the next step.

Step 3: Continue row reduction to obtain the **reduced echelon form**.

Step 4: Write the **system of equations** corresponding to the matrix obtained in step 3.

Step 5: **Rewrite** each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

solve

$$\underbrace{\begin{pmatrix} 1 & -2 & -1 & 3 \\ -2 & 4 & 5 & -5 \\ 3 & -6 & -6 & 8 \end{pmatrix}}_{3\times 4}\left|\begin{matrix} 0 \\ 3 \\ -3 \end{matrix}\right.$$

$3 \times 4$: 4 variables, 3 equations.

$$\xrightarrow[-2 \to 0]{R_2 = R_2 + 2R_1} \begin{pmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 3 & -6 & -6 & 8 & -3 \end{pmatrix} \xrightarrow[3 \to 0]{R_3 = R_3 - 3R_1} \begin{pmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & -3 & -1 & -3 \end{pmatrix}$$

$$\xrightarrow[-3 \to 0]{R_3 = R_3 + R_2} \left(\begin{array}{cccc|c} \boxed{1} & -2 & -1 & 3 & 0 \\ 0 & 0 & \boxed{3} & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$

$x_2$, $x_4$ free variable.

$x_2$ & $x_4$ are free, set

$$\begin{cases} x_2 = s \\ x_4 = t \end{cases}$$

from row 2, $3x_3 + x_4 = 3 \Rightarrow x_3 = \frac{1}{3}(3 - x_4) = 1 - \frac{t}{3}$

from row 1, $x_1 - 2x_2 - x_3 + 3x_4 = 0$

$$\Rightarrow x_1 = 2s + 1 - \frac{t}{3} - 3t = 2s - \frac{10}{3}t + 1$$

$$\begin{cases} x_1 = 2s - \frac{10}{3}t + 1 \\ x_2 = s \\ x_3 = 1 - t/3 \\ x_4 = t \end{cases}$$