Form 1, system of linear equation or augmented matrix

Section 1.3 Vector Equations

Vectors and $\mathbb{R}^2$

$\begin{pmatrix} 1 \\ 2 \end{pmatrix}_{2 \cdot 1}$

1. A matrix with only one column is a column vector or a vector.
2. The set of all vectors with two entries is denoted by $\mathbb{R}^2$.
3. Two vectors in $\mathbb{R}^2$ are equal if and only if corresponding entries are equal.
4. Sum of two vectors **u** and **v**: $\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}$.
5. Scalar multiple of **u** ($\vec{u}$) by c: $c\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ca \\ cb \end{pmatrix}$

$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$a = 1$
$b = 2$

Example 1: Given $\mathbf{u} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ $\mathbf{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ Find $2\mathbf{u}$ and $\frac{1}{2}\mathbf{u} - 2\mathbf{v}$.

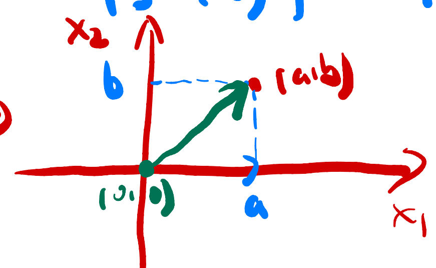
$$2\vec{u} = 2\begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 \\ 12 \end{pmatrix}$$

$$\frac{1}{2}\vec{u} - 2\vec{v} = \frac{1}{2}\begin{pmatrix} 2 \\ 6 \end{pmatrix} - 2\begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ -6 \end{pmatrix} = \begin{pmatrix} 1-2 \\ 3-(-6) \end{pmatrix} = \begin{pmatrix} -1 \\ 9 \end{pmatrix}$$

Geometric Descriptions of $\mathbb{R}^2$

$[a, b]$

We can identify a geometric point (a,b) with the column vector $\begin{bmatrix} a \\ b \end{bmatrix}$



1. Parallelogram Rule for Addition: If **u** and **v** in $\mathbb{R}^2$ are represented as points in the plane, then $\mathbf{u}+\mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are **u**, 0, and **v**. See Figure 1.

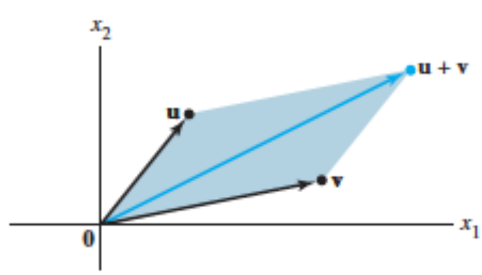
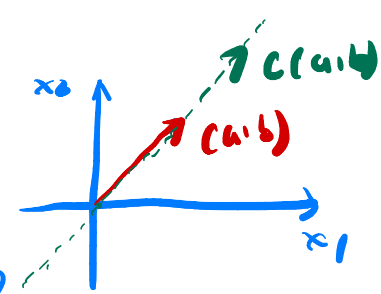


Figure 1: Parallelogram Rule



2. All scalar multiples of one fixed nonzero vector is a line through the origin, $(0,0)$.

Generalization to $\mathbb{R}^3$ and $\mathbb{R}^n$

$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}_{3\cdot 1}$

1. Vectors in $\mathbb{R}^3$ are 3×1 column matrices with three entries.
2. Let n be a positive integer, $\mathbb{R}^n$ denotes the collection of all lists of n real numbers, usually written as $n \times 1$ column matrices, such as

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \cdots \\ u_n \end{bmatrix}_{n\cdot 1}$$

n rows

Algebraic Properties of $\mathbb{R}^n$

For all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^n$ and all scalars c and d:

(i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(iii) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(iv) $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$, where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$

(v) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(vi) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(vii) $c(d\mathbf{u}) = (cd)(\mathbf{u})$

(viii) $1\mathbf{u} = \mathbf{u}$

Linear Combinations

$\begin{pmatrix} \quad \end{pmatrix}_{n\cdot 1}$ matrix

Given vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, \cdots, c_p$, the vector $\mathbf{y}$ defined by

$$\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

is called a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$ with weights $c_1, c_2, \cdots, c_p$.

Example 3: Determine if $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{a}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} -2 \\ 1 \\ 6 \end{bmatrix}$$

If $\vec{b}$ is a linear combination of $\vec{a_1}$ $\vec{a_2}$ $\vec{a_3}$

def of the linear combination.

$\Leftrightarrow$ there exist c_1 c_2 c_3 such that.

$$c_1 \vec{a_1} + c_2 \vec{a_2} + c_3 \vec{a_3} = b$$

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} c_1 + c_3 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} c_1 + c_3 = -2 \\ c_2 = 1 \\ c_3 = 6 \end{cases} \quad (*)$$

$$\Leftrightarrow \begin{cases} c_3 = 6 \\ c_2 = 1 \\ c_1 = -8 \end{cases}$$

$(*)$ has unqiue solt^{n}

$\Rightarrow$ b is a linear combination of a_1 a_2 a_3

you can find x_1 x_2 ... x_n such that (*) is true.

Theorem
A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \quad (*)$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix} \tag{1}$$

In particular, $\mathbf{b}$ can be generated by a linear combination of $\mathbf{a}_1, \cdots, \mathbf{a}_n$ if and only if there exists a solution to the linear system corresponding to the matrix (1).

Definition:
If $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$ in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \cdots, \mathbf{v}_p$ is denoted by Span$\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$and is called the subset of $\mathbb{R}^n$ spanned (or generated) by $\mathbf{v}_1, \cdots, \mathbf{v}_p$. That is, Span $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

witd $c_1, c_2, \cdots, c_p$ scalars.

Example 4: Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ $\mathbf{a}_2 = \begin{bmatrix} -5 \\ -8 \\ 2 \end{bmatrix}$ $\mathbf{b} = \begin{bmatrix} 3 \\ -5 \\ h \end{bmatrix}$. For what value(s) of h is $\mathbf{b}$ in the plane spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$?

If b is in the span$\{a_1\ a_2\}$

defination of span $\Leftrightarrow$ there exist c_1 & c_2 such that.

$$\vec{b} = c_1\vec{a_1} + c_2\vec{a_2} \quad (**)$$

by thm: to find the solution of $(**)$ $\Leftrightarrow$ to find the solution of the linear system whose augmented matrix is

$$[\vec{a_1} \quad \vec{a_2} \quad \vec{b}]$$

$$\Rightarrow \left(\begin{array}{cc|c} 1 & -5 & 3 \\ 3 & -8 & -5 \\ -1 & 2 & h \end{array}\right)$$

$\underbrace{}_{3\times 2 \text{ matrix}}$ $\Rightarrow$ 3 equations & 2 variables c_1 & c_2.

$$\xrightarrow[3 \to 0]{R_2 = R_2 - 3R_1} \begin{pmatrix} 1 & -5 & 3 \\ 0 & 7 & -14 \\ -1 & 2 & h \end{pmatrix} \xrightarrow[-1 \to 0]{R_3 = R_3 + R_1} \begin{pmatrix} 1 & -5 & 3 \\ 0 & 7 & -14 \\ 0 & -3 & 3+h \end{pmatrix}$$

$$\xrightarrow[-3 \to 0]{R_3 = R_3 + R_2\frac{3}{7}} \begin{pmatrix} 1 & -5 & 3 \\ 0 & 7 & -14 \\ 0 & 0 & 3+h \\ & & +(-14)\cdot\frac{3}{7} \end{pmatrix} = \begin{pmatrix} 1 & -5 & 3 \\ 0 & 7 & -14 \\ 0 & 0 & \underline{h-3} \end{pmatrix}$$

unknown
$0 \quad 0 \quad 1$

$\Leftrightarrow \; 0\cdot x_1 + 0\cdot x_2 = 1$ X

by the thm in 4.2

$$h - 3 = 0$$

$$\Leftrightarrow \quad h = 3.$$

A Geometric Description of Span$\{\mathbf{v}\}$ and Span$\{\mathbf{u},\mathbf{v}\}$

1. Let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^3$, Span$\{\mathbf{v}\}$ is the set of points on the line in $\mathbb{R}^3$ through $\mathbf{v}$ and $\mathbf{0}$.
2. Let $\mathbf{u}$ and $\mathbf{v}$ be a nonzero vectors in $\mathbb{R}^3$, and $\mathbf{v}$ is not a multiple of $\mathbf{u}$, then $\{\mathbf{u},\mathbf{v}\}$ is the plane in $\mathbb{R}^3$ that contains $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{0}$.

Example 5: Give a geometric description of Span$\{\mathbf{v}_1,\mathbf{v}_2\}$ for the vectors $\mathbf{v}_1 = \begin{bmatrix} 8 \\ 2 \\ -6 \end{bmatrix}$ $\mathbf{v}_2 = \begin{bmatrix} 12 \\ 3 \\ -9 \end{bmatrix}$.