Review

x1 x2 x3 variables

suppose $\left(\begin{array}{ccc|c} 1 & 4 & -5 & 0 \\ \underbrace{2 & -1 & 8}_{2\times 3} & 9 \end{array}\right)$ is an augmented matrix of a linear system. Solve it!

$[\vec{a}_1 \ \vec{a}_2, \dots, \vec{a}_n \ \vec{b}] \rightarrow$ augmented matrix

$\Leftrightarrow \quad x_1 \vec{a}_1 + x_2 \vec{a}_2 + \dots x_n \vec{a}_n = \vec{b},$ vector form

$x_1 \dots x_n$ are scalars (unknown)

$$\begin{pmatrix} 1 & 4 & -5 & 0 \\ 2 & -1 & 8 & 9 \end{pmatrix} \xrightarrow{R_2 = R_2 - 2R_1} \begin{pmatrix} 1 & 4 & -5 & 0 \\ 0 & -9 & 18 & 9 \end{pmatrix}$$

$2 \rightarrow 0$

REF

$\underbrace{x_1 \quad x_2}_{\text{basic}}$ x_3 free

x_3 is free, set $\boxed{x_3 = s}$

from the row 2, $-9x_2 + 18x_3 = 9$

$9x_2 = 18x_3 - 9 = 18s - 9$

$\boxed{x_2 = 2s - 1}$

from the row 1, $x_1 + 4x_2 - 5x_3 = 0$

$\boxed{x_1 = -4(2s-1) + 5s = -3s + 4}$

Section 1.4 The Matrix Equation $Ax = b$

Definition: Matrix vector product $A = \{\vec{a}_1, \cdots, \vec{a}_n\}$, $a_1 \dots a_n \in \mathbb{R}^m$

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \cdots, \mathbf{a}_n$, and if $\mathbf{x}$ in $\mathbb{R}^n$, then the product of A and $\mathbf{x}$, is the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights, that is

$$A x = [\vec{a}_1 \cdots \vec{a}_n] \cdot \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{x \in \mathbb{R}^n} = x_1 \vec{a}_1 + x_2 \vec{a}_2 + \cdots x_n \vec{a}_n$$

Example 1: Use the definition to compute

$$\begin{array}{ccc} \begin{bmatrix} 1 & 3 & -4 \\ 4 & 5 & 2 \end{bmatrix} & \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \\ \vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3 & \vec{x} \end{array}$$

$$Ax = \{\vec{a}_1, \vec{a}_2, \vec{a}_3\} \vec{x} \stackrel{def}{=} x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3$$

$$= 1 \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} + 2 \cdot \begin{pmatrix} 3 \\ 5 \end{pmatrix} + 1 \begin{pmatrix} -4 \\ 2 \end{pmatrix}$$

vector multiplied by a scalar

$$= \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 6 \\ 10 \end{pmatrix} + \begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 16 \end{pmatrix}$$

Row-Vector Rule for Computing $A\mathbf{x}$:

If the product $A\mathbf{x}$ is defined, then the i th entry in $A\mathbf{x}$ is the sum of the products of corresponding entries from row i of A and from the vector $\mathbf{x}$.

Example 2: Use the Row-Vector Rule to compute

$$\begin{bmatrix} 1 & 3 & -4 \\ 4 & 5 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

the first row of A

1st entry of Ax, $[1 \;\; 3 \;\; -4] \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 1 \cdot 1 + 3 \cdot 2 + (-4) \cdot 1 = 3$

2nd entry of Ax $[4 \;\; 5 \;\; 2] \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 4 \cdot 1 + 5 \cdot 2 + 2 \cdot 1 = 16$

row 2 of A.

Theorem:

If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar, then:

1. $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$
2. $A(c\mathbf{u}) = cA\mathbf{u}$ scalar

Theorem: Three equivalent ways of representing a linear system
If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \cdots, \mathbf{a}_n$, and if $\mathbf{b}$ in $\mathbb{R}^m$, the matrix equation

$$A\mathbf{x} = \mathbf{b} \tag{1}$$

the solution of (1): vector $x \in \mathbb{R}^n$ which solves $Ax = b$.

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \tag{2}$$

which has the same solution set as the system of linear equations whose augmented matrix is

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n \quad \mathbf{b}] \tag{3}$$

Example 3: Write the system first as a vector equation and then as a matrix equation.

$$5x_1 + 3x_2 - 2x_3 = 0$$
$$0 \cdot x_1 + x_2 + 6x_3 = 9$$

$\vec{a}_1$: coeff of x_1 in each equations

$\vec{a}_1 = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$

$\vec{a}_2$: coeff of x_2 in each eqn.

$\vec{a}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\vec{a}_3 = \begin{pmatrix} -2 \\ 6 \end{pmatrix}$

vector equation.

$\boxed{x_1 \cdot \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3 = \vec{b}, \quad \vec{b} = \begin{pmatrix} 0 \\ 9 \end{pmatrix}}$

$\boxed{Ax = b, \quad A = \{\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3\} = \begin{pmatrix} 5 & 3 & -2 \\ 0 & 1 & 6 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 9 \end{pmatrix}}$

matrix equation.

Example 4: Let $A = \begin{bmatrix} -3 & -4 \\ 12 & 16 \end{bmatrix}$, $\quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Show that the equation $A\mathbf{x} = \mathbf{b}$ does not have a solution for some choices of $\mathbf{b}$, and describe the set of all $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ does have a solution.

by the theorem, we can solve the augmented form,

$$[A \ \vec{b}] = \begin{bmatrix} -3 & -4 & b_1 \\ 12 & 16 & b_2 \end{bmatrix}$$

Find the solution of the system whose augmented matrix

is $[A \ \vec{b}]$

x_1 basic

x_2 is free

REF

$$\begin{bmatrix} -3 & -4 & b_1 \\ 12 & 16 & b_2 \end{bmatrix} \xrightarrow[12 \to 0]{R_2 = R_1 \cdot 4 + R_2} \begin{bmatrix} \boxed{-3} & -4 & b_1 \\ 0 & 0 & \underline{4b_1 + b_2} \end{bmatrix}$$

by theorem in 1.2, the system has a solution

if $4b_1 + b_2 = 0$.

Now set $\boxed{x_2 = s \quad (\text{free})}$

from the 1st row $-3x_1 - 4x_2 = b_1$

$$-3x_1 = +4s + b_1$$

$$\boxed{x_1 = -\frac{1}{3}(4s + b_1)}$$

Theorem: $\rightarrow A = \{\vec{a_1}, \vec{a_2}, \cdots, \vec{a_n}\}$

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent.

1. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
2. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A. $\Leftrightarrow$ for any $\vec{b} \in \mathbb{R}^m$, there exist $x_1, x_2, \cdots x_n$ such that. $\vec{b} = x_1\vec{a_1} + x_2\vec{a_2} + \cdots + x_n\vec{a_n}$
3. The columns of A span $\mathbb{R}^m$.
4. A has a pivot position in every row.

Example 5: Let $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$. Does $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ span $\mathbb{R}^3$?

this is point #3 in the theorem

so we can use point #4 to check.

$$A = \{\vec{v_1}\ \vec{v_2}\ \vec{v_3}\} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & -1 & -1 \\ -1 & 3 & -3 \end{pmatrix}$$

$$\xrightarrow[R_1 \,\&\, R_3]{\text{interchang}} \begin{pmatrix} \boxed{-1} & 3 & -3 \\ 0 & \boxed{-1} & -1 \\ 0 & 0 & \boxed{2} \end{pmatrix} \text{REF}$$

we need to check, if there is a leading entry in each row.

by the theorem, $\{v_1\ v_2\ v_3\}$ span $\mathbb{R}^3$. //