# Section 1.5 Solution sets of linear systems

const vector = 0

## Homogeneous Linear Systems:

A system of linear equations is said to be homogeneous if it can be written in the form $A\mathbf{x} = \mathbf{0}$, where $A$ is an $m \times n$ matrix, and $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$. // Such a system $A\mathbf{x} = \mathbf{0}$ always has at least one solution, namely $\mathbf{x} = \mathbf{0}$. This solution is called the trivial solution.

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \rightarrow$ homogeneous system. $\quad Ax = \vec{0}$

$= \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{\neq 0} \rightarrow$ inhomogeneous

(solution which is not equal to 0)

**Theorem:** The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

**Example 1:** Determine if the following homogeneous system

$$
\begin{aligned}
2x_1 - 5x_2 + 8x_3 &= 0 \\
-2x_1 - 7x_2 + x_3 &= 0 \\
4x_1 + 2x_2 + 7x_3 &= 0
\end{aligned}
$$

$= 0 \Rightarrow$ homog.

has a nontrivial solution.

we can go from the coeff matrix directly.

$$
\begin{pmatrix} 2 & -5 & 8 \\ -2 & -7 & 1 \\ 4 & 2 & 7 \end{pmatrix} \xrightarrow[-2 \to 0]{R_2 = R_2 + R_1} \begin{pmatrix} 2 & -5 & 8 \\ 0 & -12 & 9 \\ 4 & 2 & 7 \end{pmatrix} \xrightarrow[4 \to 0]{R_3 = R_3 - 2R_1} \begin{pmatrix} 2 & -5 & 8 \\ 0 & -12 & 9 \\ 0 & 12 & -9 \end{pmatrix}
$$

$$
\xrightarrow[12 \to 0]{R_3 = R_2 + R_3} \begin{pmatrix} 2 & -5 & 8 \\ 0 & -12 & 9 \\ 0 & 0 & 0 \end{pmatrix} \text{REF}
$$

**Example 2:** Determine if the following homogeneous system

$$2x_1 - x_2 - 3x_3 = 0$$

has a nontrivial solution.

$$\begin{array}{ccc} \multicolumn{3}{c}{\begin{pmatrix} \boxed{2} & -5 & 8 \\ 0 & \boxed{-12} & 9 \\ 0 & 0 & 0 \end{pmatrix}} \\ \underbrace{x_1 \quad x_2}_{\text{basic}} & & x_3 \text{ is free} \end{array}$$

set $x_3 = s$, $s$ is the parameter & it's free

from the 2nd row $\quad -12x_2 + 9x_3 = 0$

$$12x_2 = 9s$$

$$x_2 = \frac{3}{4}s$$

from the 1st row $\quad 2x_1 - 5x_2 + 8x_3 = 0$

$$2x_1 = \left(\frac{15}{4}s - 8s\right)$$

$$x_1 = \left(\frac{15}{8} - \frac{32}{8}\right)s = \frac{-17}{8}s$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{17}{8}s \\ 3/4\, s \\ s \end{pmatrix} = s \cdot \begin{pmatrix} -\frac{17}{8} \\ 3/4 \\ 1 \end{pmatrix},$$

$s$ is any number.

**Parametric Vector Form:**
Whenever a solution set is described explicitly with vectors as in Example 1 or 2,

$$\mathbf{x} = s\mathbf{u} + t\mathbf{v}, \quad s,t \in \mathbb{R}$$

we say that the solution is in parametric vector form.
Sometimes, the free parameters are denoted by $s$, $t$ etc. as to emphasize that the parameters vary over all real numbers.

**Solutions of Nonhomogeneous Systems:**
When a nonhomogeneous linear system has many solutions, the general solution can be written in parametric vector form as one vector plus an arbitrary linear combination of vectors that satisfy the corresponding homogeneous system.

**Example 4:** Describe the solutions of the following system in parametric vector form.

$$\begin{aligned} 2x_1 + 2x_2 + 4x_3 &= 8 \\ -4x_1 - 4x_2 - 8x_3 &= -16 \\ -3x_2 - 3x_3 &= 12 \end{aligned}$$

A geometric description:

**Theorem :** Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

**Summary:** Writing a solution set (of a consistent system) in parametric vector form

1. Row reduce the augmented matrix to reduced echelon form.
2. Express each basic variable in terms of any free variables appearing in an equation.
3. Write a typical solution $\mathbf{x}$ as a vector whose entries depend on the free variables, if any.
4. Decompose x into a linear combination of vectors (with numeric entries) using the free variables as parameters.