Section 1.7 Linear Independence

Definition:
An indexed set of vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$ in $\mathbb{R}^n$ is said to be linearly independent if the vector equation

$$x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. $\Leftrightarrow$ the system has no free variable.
The set $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$ is said to be linearly dependent if there exist weights $c_1, c_2, \cdots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

$\Leftrightarrow$ the vector equation has non trivial solutions $\Leftrightarrow$ the system has at least one free variables

$\Leftrightarrow$ there exists at least one c_i, $c_i \neq 0$

Example 1: Determine if the vectors are linearly independent $\mathbf{v}_1 = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$ $\mathbf{v}_2 = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}$ $\mathbf{v}_3 = \begin{bmatrix} 9 \\ 4 \\ -8 \end{bmatrix}$

$$\{v_1\ v_2\ v_3\} = \begin{pmatrix} 5 & 7 & 9 \\ 0 & 2 & 4 \\ 0 & -6 & -8 \end{pmatrix} \xrightarrow[-6 \to 0]{R_3 = R_3 + 3R_2} \begin{pmatrix} \boxed{5} & 7 & 9 \\ 0 & \boxed{2} & 4 \\ 0 & 0 & \boxed{4} \end{pmatrix}$$

$x_1\ x_2\ x_3$ are basic $\Leftrightarrow$ there is no free variable $\Leftrightarrow$ trivial solution only

$\Leftrightarrow$ linearly indep.

Example 2: Find the value(s) of h for which the vectors are linearly dependent. $\begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}$ $\begin{bmatrix} -2 \\ -9 \\ 6 \end{bmatrix}$ $\begin{bmatrix} 3 \\ h \\ -9 \end{bmatrix}$

$$\begin{pmatrix} 1 & -2 & 3 \\ 5 & -9 & h \\ -3 & 6 & -9 \end{pmatrix} \xrightarrow[5 \to 0]{R_2 = R_2 - 5R_1} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & h-15 \\ -3 & 6 & -9 \end{pmatrix} \xrightarrow[-3 \to 0]{R_3 = R_3 + 3R_1} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & h-15 \\ 0 & 0 & 0 \end{pmatrix}$$

REF

x_3 free

because x_3 is free $\Leftrightarrow$ we have non-trivial solutions

$\Leftrightarrow$ system is linearly dep.

$A = \{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n\}_{m \times n}$

Linear independence of matrix columns:

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \cdots, \mathbf{a}_n$, then the matrix equation $A\mathbf{x} = \mathbf{0}$ can be written as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$$

Then the columns of a matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

x1 x2 x3 variables

Example 3: Let the matrix $\underbrace{\begin{bmatrix} 0 & -3 & 9 & 0 \\ 2 & 1 & -7 & 0 \\ -1 & 4 & -5 & 0 \\ 1 & -4 & -2 & 0 \end{bmatrix}}_{4 \times 3}$ be an augmented matrix of a linear system, and it can be row reduced to $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, does the coefficient matrix A have linear independent columns?

$x_1 \; x_2 \; x_3$

$\Downarrow$

they are all basic variables.

$\Leftrightarrow$ trivial solution only

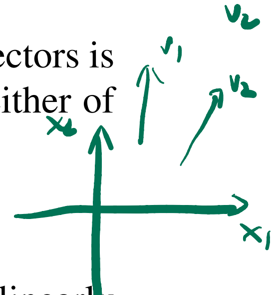
$\Leftrightarrow$ linearly indep.

Set of one or two vectors:

1. A set containing only one vector $\mathbf{v}$ is linearly independent if and only if $\mathbf{v}$ is not the zero vector. $\rightarrow \{\vec{v}\} \rightarrow$ $\vec{v} = 0$ linearly dep. / $\vec{v} \neq 0$ linearly indep.
2. A set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other. The set is linearly independent if and only if neither of the vectors is a multiple of the other.

dependent: there exists constant α such that $v_1 = \alpha v_2$

indep. $\Updownarrow$ v_1 is not $\parallel$ v_2



Set of two or more vectors:

1. Theorem: An indexed set $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others.

   In fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \cdots, \mathbf{v}_{j-1}$.

   Remark:

$(| \ | \ . | \cdots \ |)$ p: # of vectors in the set

n: the size/dim of the vectors in the set.

2. Theorem: If a set contains more vectors than the number of entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$. // But when $n \geq p$, we can not see this is linearly indep/dep.

3. Theorem: If a set $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

Example 4: Determine whether the vectors are linearly independent. $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 2 \\ 8 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

3 vectors, $p = 3$

$n = 2$ $\quad p > 2$ by thm 2 $\Rightarrow$ linearly dep.

$$\begin{pmatrix} 5 & 2 & 1 \\ 1 & 8 & 3 \end{pmatrix} \xrightarrow[1 \to 0]{R_2 = R_2 - R_1 \cdot \frac{1}{5}}$$

$$\xrightarrow[R_1 \,\&\, R_2]{\text{interchange}} \begin{pmatrix} 1 & 8 & 3 \\ 5 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow[5 \to 0]{R_2 = R_2 - 5R_1} \begin{pmatrix} \boxed{1} & 8 & 3 \\ 0 & \boxed{-38} & -14 \end{pmatrix}$$

x_3 is free

$\Leftrightarrow$ the system has non-trivial solutions

$\Leftrightarrow$ linear dependent.