Section 1.8 Introduction to Linear Transformations

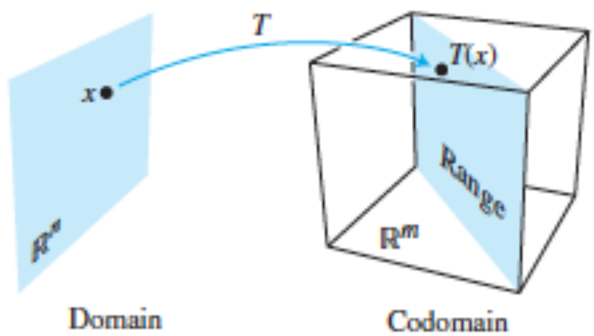
A matrix equation $A\mathbf{x} = \mathbf{b}$ can be thought of the matrix A as an object that "acts" on a vector $\mathbf{x}$ by multiplication to produce a new vector called $A\mathbf{x}$.

Definition: A transformation (map) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$. Denote by $T : \mathbb{R}^n \to \mathbb{R}^m$.

The set $\mathbb{R}^n$ is called the domain of T, and $\mathbb{R}^m$ is called the codomain of T. For $\mathbf{x}$ in $\mathbb{R}^n$, $T(\mathbf{x})$ is called the image of $\mathbf{x}$. The set of all images $T(\mathbf{x})$ is called the range of T.

all the possible input of T

the set which contains the images of all x in the domain.



Example 1: Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, and define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$. Find the images under T of $\mathbf{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$.

the 1st row of A

vector

$\vec{u}$, $T(\vec{u}) \overset{def}{=} A\vec{u} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}_{2\cdot 2} \cdot \begin{pmatrix} 1 \\ -3 \end{pmatrix}_{2\cdot 1} = \begin{pmatrix} \boxed{2}\,\boxed{1} + \boxed{0}\,\boxed{-3} \\ \boxed{0}\,1 + \boxed{2}\,{-3} \end{pmatrix}_{2\cdot 1}$

the 2nd row of A

$\vec{v}$, $T(\vec{v}) = A\cdot\vec{v} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{pmatrix} 2\cdot a + 0\cdot b \\ 0\cdot a + 2\cdot b \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \end{pmatrix}$

Remark: If A is an $n \times m$ matrix, solving the equation $A\mathbf{x} = \mathbf{b}$ amounts to finding all vectors $\mathbf{x}$ in $\mathbb{R}^n$ that are transformed into the vector $\mathbf{b}$ in $\mathbb{R}^m$ under the "action" of multiplication by A.

Example 3: The trasnformation T defined by $T(\mathbf{x}) = A\mathbf{x}$, find a vector $\mathbf{x}$ whose image under T is $\mathbf{b}$, and determine whether $\mathbf{x}$ is unique.

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & -5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix}$$

Target: find x such that.

$$A x = \vec{b}$$

$$[A \ \vec{b}] = \begin{pmatrix} 1 & 0 & -2 & -1 \\ -2 & 1 & 6 & 7 \\ 3 & -2 & -5 & -3 \end{pmatrix}$$

$$\xrightarrow[-2 \to 0]{R_2 = R_2 + 2R_1} \begin{pmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 3 & -2 & -5 & -3 \end{pmatrix}$$

$$\xrightarrow[3 \to 0]{R_3 = R_3 - 3R_1} \begin{pmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & -2 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow[-2 \to 0]{R_3 = R_2 \times 2 + R_3} \begin{array}{c} \begin{array}{cccc} \nearrow x_1 & \nearrow x_2 & \nearrow x_3 & \end{array} \\ \begin{pmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 5 & 10 \end{pmatrix} \end{array} \text{ REF}$$

Since we don't have a row $[0\ 0\ 0\ b]$ $b \neq 0$, $\Rightarrow$ solution exists.

Since we don't a free variable

$\Rightarrow$ solution is unique.

Solution of the system.

"go from the last row"

3rd $5x_3 = 10 \Rightarrow x_3 = 2$

2nd $x_2 + 2x_3 = 5$

$x_2 = 5 - 4 = 1$

1st $x_1 - 2x_3 = -1$

$x_1 = 3$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

Linear Transformations:

Definition: A transformation T is linear if:

1 $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}))$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T,

2 $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all $\mathbf{u}$ in the domain of T.

Facts: If T is a linear transformation, then

1 $T(\mathbf{0}) = \mathbf{0}$

2 $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$

$T(c_1\vec{v_1} + c_2\vec{v_2} + \cdots c_n\vec{v_n}) = c_1 T(\vec{v_1}) + c_2 T(\vec{v_2}) + \cdots c_n T(\vec{v_n})$

for all scalars c, d and all $\mathbf{u}, \mathbf{v}$ in the domain of T.

Example 4: Let $\mathbf{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}$, $\mathbf{y}_1 = \begin{bmatrix}2\\5\end{bmatrix}$, and $\mathbf{y}_2 = \begin{bmatrix}-1\\6\end{bmatrix}$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{e}_1$ into $\mathbf{y}_1$ and maps $\mathbf{e}_2$ into $\mathbf{y}_2$. Find the images of $\vec{v} = \begin{bmatrix}5\\-3\end{bmatrix}$

$T(\vec{e}_1) = \vec{y}_1 \quad T(\vec{e}_2) = \vec{y}_2$

$\vec{v} = \begin{pmatrix}5\\-3\end{pmatrix} = \begin{pmatrix}5\\0\end{pmatrix} + \begin{pmatrix}0\\-3\end{pmatrix}$

$= 5\begin{pmatrix}1\\0\end{pmatrix} + (-3)\begin{pmatrix}0\\1\end{pmatrix}$

$= 5\vec{e}_1 - 3\vec{e}_2$

solve the vector equation $x_1\vec{e}_1 + x_2\vec{e}_2 = \vec{v}$

$T(\vec{v}) = T(5\vec{e}_1 - 3\vec{e}_2)$

$\overset{\text{Fact \#2}}{=} 5T(\vec{e}_1) - 3T(\vec{e}_2)$

$= 5\vec{y}_1 - 3\vec{y}_2$

$= 5\begin{pmatrix}2\\5\end{pmatrix} - 3\begin{pmatrix}-1\\6\end{pmatrix} = \begin{pmatrix}13\\7\end{pmatrix}$

Example 5: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a linearly dependent set in $\mathbb{R}^n$. Explain why the set $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly dependent.

If $\vec{v}_1\ \vec{v}_2\ \vec{v}_3$ are linear dep.

there exists $c_1\ c_2\ c_3$ not all zero. such that.

$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = 0$

$T(\underbrace{c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3}_{0}) = 0$

$\overset{\text{Facts \#2}}{=} \underbrace{c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2) + c_3 T(\vec{v}_3)}$ $\;\| \; 0$

by the definition of the linear dependency

$\Rightarrow \{T(\vec{v}_1)\ T(\vec{v}_2)\ T(\vec{v}_3)\}$ are linearly dep.