Section 1.9 The matrix of a linear transformations

Example 1: The columns of $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Suppose T is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$ such that $T(e_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix}$ and $T(e_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$. With no additional information, find a formula for the image of an arbitrary $\mathbf{x}$ in $\mathbb{R}^2$.

$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. x_1 x_2 are arbitrary in $\mathbb{R}$. (scalar)

$$T(\vec{x}) = T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \end{bmatrix}\right)$$

$$\overset{\text{scalars}}{=} T\Big(x_1 \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{e}_1} + x_2 \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{e}_2}\Big)$$

$$\overset{\text{Facts \#2}}{=} x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2)$$

$$= x_1 \begin{pmatrix} 5 \\ -7 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 8 \\ 0 \end{pmatrix}$$

$\vec{v}_1$, $\vec{v}_2$

$T(\vec{v}_1) = \vec{y}_1$

$T(\vec{v}_2) = \vec{y}_2$

$T(\vec{w})$

$\vec{w} = c_1 \vec{v}_1 + c_2 \vec{v}_2$

solve for c_1 & c_2

$[\vec{v}_1 \ \ \vec{v}_2] \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \vec{w}$

$T(\vec{w}) = c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2)$

Theorem: #1 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x}, \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

$A = [\vec{a}_1 \ \vec{a}_2 \dots \vec{a}_n]_{m \cdot n}$.

In fact, A is the $m \times n$ matrix whose jth column is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the jth column of the identity matix in $\mathbb{R}^n$:

$$A = [T(\vec{\mathbf{e}}_1) \quad \cdots \quad T(\vec{\mathbf{e}}_n)] \tag{1}$$

The matrix A in (1) is called the standard matrix for the linear transformation T.

$\vec{e}_j \in \mathbb{R}^n$,

$\vec{e}_j = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow$ jth entry this the only entry $\neq 0$,

$\vec{e}_2 \in \mathbb{R}^3$ $\quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_{3 \cdot 1}$

$\vec{e}_2 \in \mathbb{R}^4$ $\quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}_{4 \cdot 1}$

Example 2:

(1)Find the standard matrix A for $T:\mathbb{R}^2 \to \mathbb{R}^2$ is a vertical shear transformation that maps $\mathbf{e}_1$ into $\mathbf{e}_1 - 3\mathbf{e}_2$, but leaves $\mathbf{e}_2$ unchanged.

$T(\vec{e_1}) = \vec{e_1} - 3\vec{e_2}$

$T(\vec{e_2}) = \vec{e_2}$

$A \overset{\text{thm}}{=} [T(\vec{e_1}), T(\vec{e_2})]_{2\cdot 2}$

the first col the 2nd col

$= [\vec{e_1} - 3\vec{e_2}, \vec{e_2}]$

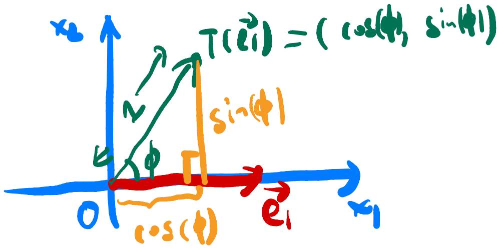
$= \left[\begin{pmatrix}1\\0\end{pmatrix} - 3\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right]$

$= \begin{bmatrix}1 & 0\\-3 & 1\end{bmatrix} = A$

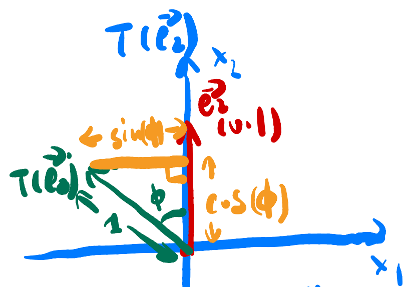
(2) Find the standard matrix A for the rotation transformation that rotates each point $\mathbb{R}^2$ about the origin through an anlge ϕ. (counter-clockwise)

step 1, $\vec{e_1} = \begin{pmatrix}1\\0\end{pmatrix}$ $\vec{e_2} = \begin{pmatrix}0\\1\end{pmatrix}$

step 2, $T(\vec{e_1})$



$T(\vec{e_1}) = \begin{pmatrix}\cos(\phi)\\\sin(\phi)\end{pmatrix}$



$\Rightarrow T(\vec{e_2}) = \begin{pmatrix}-\sin(\phi)\\\cos(\phi)\end{pmatrix}$

step 3.

$A = [T(\vec{e_1}), T(\vec{e_2})]$

$= \begin{pmatrix}\cos(\phi), & -\sin(\phi)\\\sin(\phi), & \cos(\phi)\end{pmatrix}$.

Geometric interpretations: See Table 1-4.

Example 3: Assume that T is a linear transformation. Find the standard matrix of $T:\mathbb{R}^2 \to \mathbb{R}^2$ first reflects points through the ~~line $x_2 = x_1$~~ x_1 axis and then reflects points through the line $x_2 = x_1$.

T_1 T_2

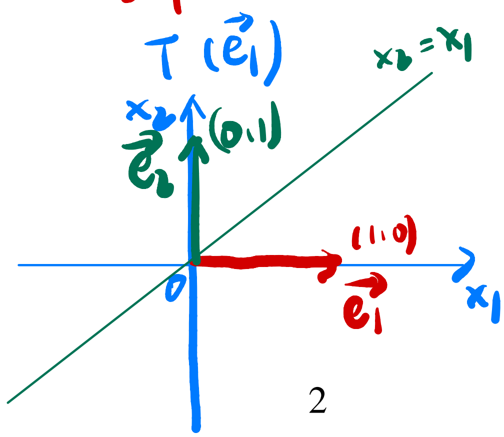
step 1

$\vec{e_1} = \begin{pmatrix}1\\0\end{pmatrix}$

$\vec{e_2} = \begin{pmatrix}0\\1\end{pmatrix}$

step 2.

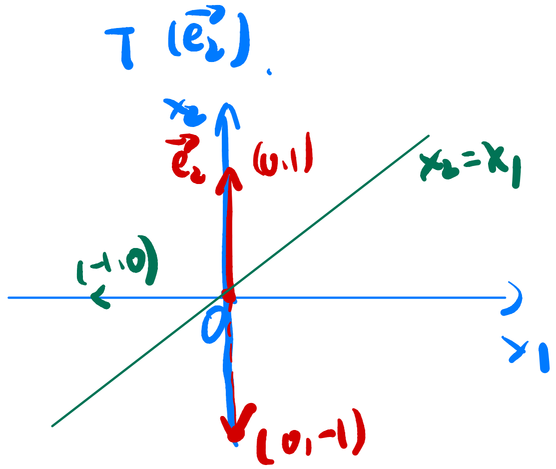
$T(\vec{e_1})$



$T_1(\vec{e_1}) = \vec{e_1}$

$T_2(\vec{e_1}) = \vec{e_2}$

$T(\vec{e_1}) = \vec{e_2} = \begin{pmatrix}0\\1\end{pmatrix}$



$T_1(\vec{e_2}) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -\vec{e_2}$

$T_2(-\vec{e_2}) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -\vec{e_1}$

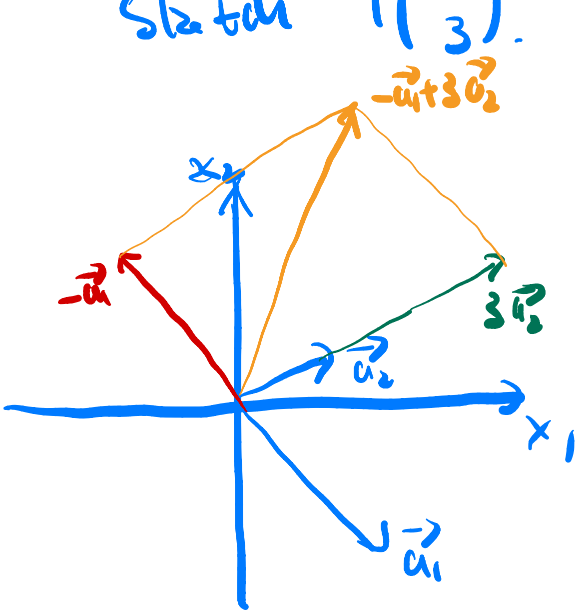
std matrix : $A = [T(\vec{e_1}), T(\vec{e_2})]$

$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Q4 Let T be a linear transformation with the std matrix $\underline{A = [\vec{a_1}, \vec{a_2}]}$ as shown in the figure.

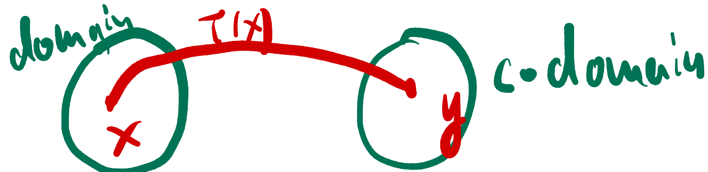
$\vec{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Sketch $T\begin{pmatrix} -1 \\ 3 \end{pmatrix}$.



$T\begin{pmatrix} -1 \\ 3 \end{pmatrix} = T\left(\begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix}\right) = T(-\vec{e_1} + 3\vec{e_2})$

$\overset{\text{Facts \#2}}{\underset{1.8}{=}} (-1) \cdot T(\vec{e_1}) + 3 \cdot T(\vec{e_2}) \overset{\text{thm \#1}}{=} -\vec{a_1} + 3\vec{a_2}$



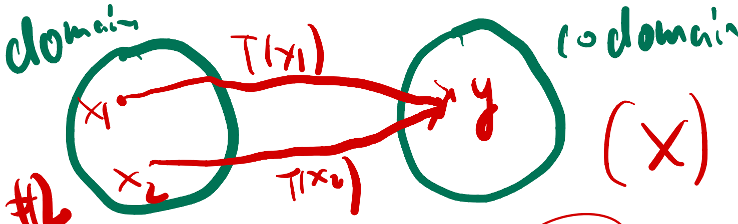
Definition:

1. A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at least one $\mathbf{x}$ in $\mathbb{R}^n$. This is an existense question. (surjective)

for any $\vec{y} \in \mathbb{R}^m$, we can find at least one $\vec{x} \in \mathbb{R}^n$

such that $\vec{y} = T(\vec{x})$

2. A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is one to one if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at most one $\mathbf{x}$ in $\mathbb{R}^n$. This is a uniqueness question. (injective)



(X) not one-to-one.

#2

Theorem: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

$(\Leftrightarrow)$

there exists $(\exists)$ A such that (s.t.) $Ax = T(x)$

$Ax = 0$ has only trivial solution.

#3

$\Leftrightarrow$ no free variable.

Theorem: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then:

(check 1.4)

1. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$.

2. T is one-to-one if and only if the columns of A are linearly independent.

$A = [\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n]$ $\quad x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n = 0$

Example 4: Let $T(x_1,x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$. Show that T is a one-to-one linear transformation.

$\downarrow$ has only trivial solution.

$$T\left(\begin{pmatrix}x_1\\x_2\end{pmatrix}\right) = \begin{pmatrix}3x_1+x_2\\5x_1+7x_2\\x_1+3x_2\end{pmatrix} = \begin{pmatrix}3x_1\\5x_1\\x_1\end{pmatrix} + \begin{pmatrix}x_2\\7x_2\\3x_2\end{pmatrix}$$

$$= x_1 \underbrace{\begin{pmatrix}3\\5\\1\end{pmatrix}}_{\vec{a}_1} + x_2 \underbrace{\begin{pmatrix}1\\7\\3\end{pmatrix}}_{\vec{a}_2}$$

we need show

$x_1 \vec{a}_1 + x_2 \vec{a}_2 = 0$ has only trivial solution.

$$\begin{pmatrix}\vec{a}_1 & \vec{a}_2\end{pmatrix} = \begin{pmatrix}3 & 1\\5 & 7\\1 & 3\end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix}1 & 3\\5 & 7\\3 & 1\end{pmatrix}$$

$$\xrightarrow[5 \to 0]{R_2 = R_2 - 5R_1} \begin{pmatrix}1 & 3\\0 & -8\\3 & 1\end{pmatrix} \xrightarrow[3 \to 0]{R_3 = R_3 - 3R_1} \begin{pmatrix}1 & 3\\0 & -8\\0 & -8\end{pmatrix} \xrightarrow[-8 \to 0]{R_3 = R_3 - R_2} \overset{\text{REF}}{\begin{pmatrix}1 & 3\\0 & -8\\0 & 0\end{pmatrix}}$$

$$\begin{array}{cc} x_1 & x_2 \\ \checkmark & \end{array}$$

x_1 & x_2 are basic

$\Rightarrow$ only trivial solution.

TABLE 1 Reflections

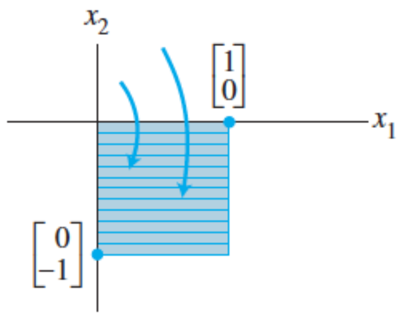
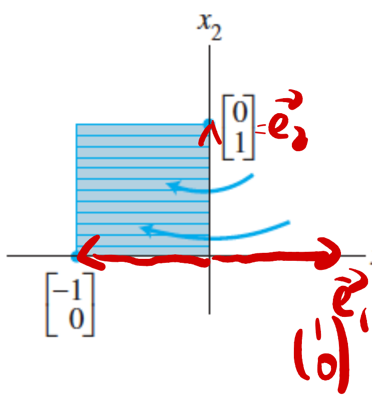
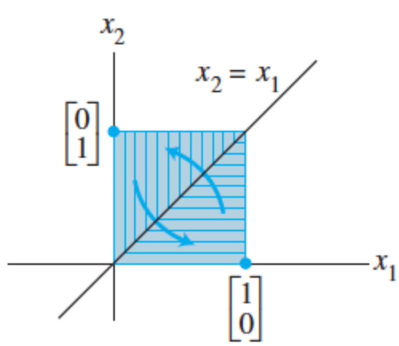
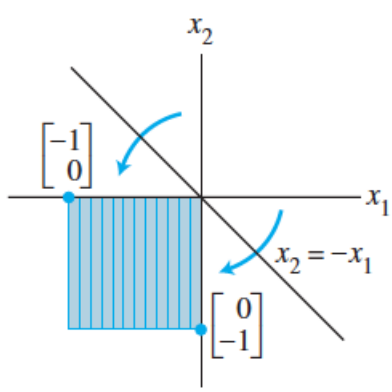
Transformation	Image of the Unit Square	Standard Matrix
Reflection through the x_1-axis		$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$
Reflection through the x_2-axis		$\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$
Reflection through the line $x_2 = x_1$		$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$
Reflection through the line $x_2 = -x_1$		$\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}$

TABLE 2 Contractions and Expansions

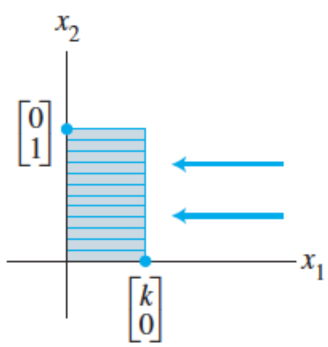
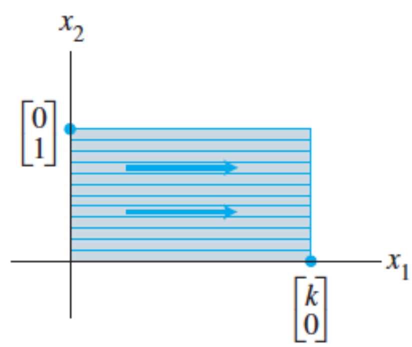
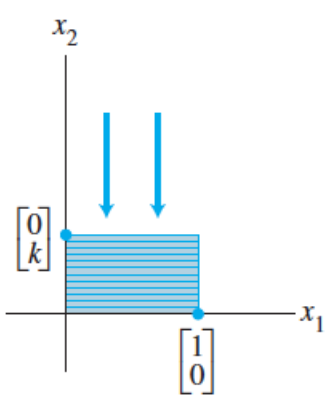
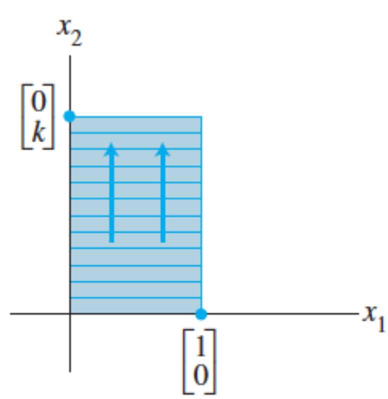
Transformation	Image of the Unit Square		Standard Matrix
Horizontal contraction and expansion	 $0 < k < 1$	 $k > 1$	$\begin{bmatrix}k & 0\\0 & 1\end{bmatrix}$
Vertical contraction and expansion	 $0 < k < 1$	 $k > 1$	$\begin{bmatrix}1 & 0\\0 & k\end{bmatrix}$

TABLE 3 Shears

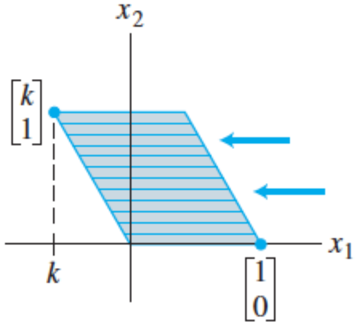
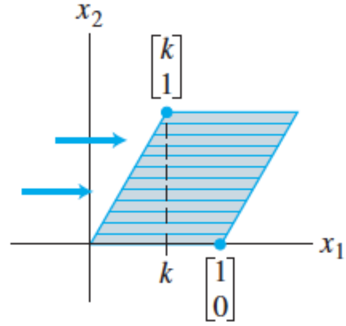
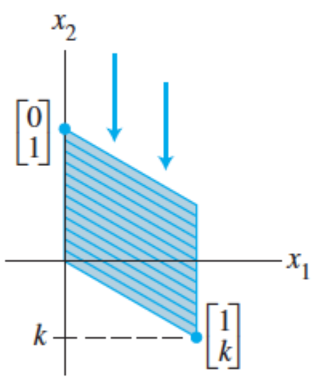
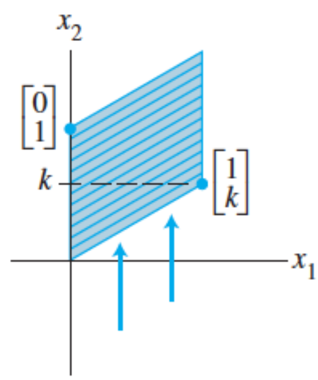
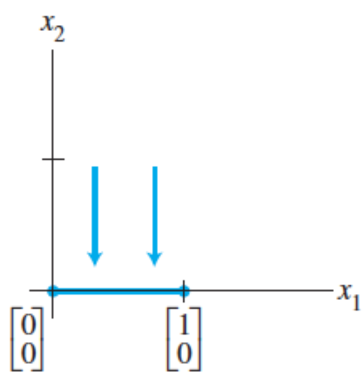
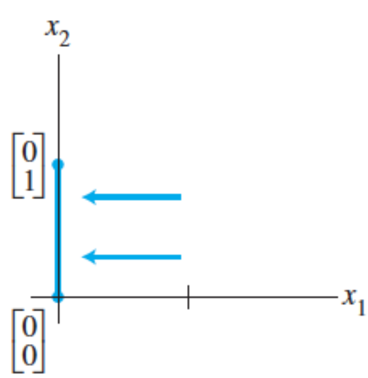
Transformation	Image of the Unit Square		Standard Matrix
Horizontal shear	 $k < 0$	 $k > 0$	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Vertical shear	 $k < 0$	 $k > 0$	$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

TABLE 4 Projections

Transformation	Image of the Unit Square	Standard Matrix
Projection onto the x_1-axis		$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$
Projection onto the x_2-axis		$\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$