$a_{ij} = A_{ij}$ i-th row j-th col entry of the matrix

Section 2.1 Matrix Operations

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $A_{ij} = a_{12} = 2 = 2$ 2nd col, 1st row

Definitions:

a_{11} $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ a_{22}

1. The diagonal entries in an $m \times n$ matrix $A = [a_{ij}]$ are $a_{11}, a_{22}, \cdots$, and they form the main diagonal of A. (row index = col index ⇒ diagonal entry)
2. A diagonal matrix is a $\boxed{\text{square } n \times n \text{ matrix}}$ whose nondiagonal entris are zero.
   # rows = # cols ⇒ square matrix

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3\cdot 3}$

3. An $m \times n$ matrix whose entries are all zero is a zero matrix and is written as 0.
4. Two matrices are equal if they have the ① same size and if their corresponding entries are equal.
   # rows are the same
   # cols - - - -

Sums and Scalar Multiples:

1. If A and B are $m \times n$ (same size) matrices, then the sum $A+B$ is the $m \times n$ matrix whose columns are the sums of the corresponding columns in A and B.
2. If r is a scalar and A is a matrix, then the scalar multiple rA is the matrix whose columns are r times the corresponding columns in A.

Theorem: Let A, B and C be matrices of the same size, and let r and s be scalars.

$$A + B = B + A \qquad r(A+B) = rA + rB$$
$$(A+B)+C = A+(B+C) \qquad (r+s)A = rA + sA$$
$$A + 0 = A \qquad r(sA) = rsA$$

Example 1: Let $A = \begin{bmatrix} 2 & 0 \\ 4 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & -5 \\ 1 & -4 \end{bmatrix}$, compute $-3A$ and $A - 2B$.

$$-3A = -3 \cdot \begin{pmatrix} 2 & 0 \\ 4 & -5 \end{pmatrix} = \begin{pmatrix} -3\cdot 2, & -3\cdot 0 \\ -3\cdot 4 & -3\cdot -5 \end{pmatrix} = \begin{pmatrix} -6 & 0 \\ -12 & 15 \end{pmatrix}$$

$$A - 2B = \begin{pmatrix} 2 & 0 \\ 4 & -5 \end{pmatrix} - 2\begin{pmatrix} 7 & -5 \\ 1 & -4 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 4 & -5 \end{pmatrix} - \begin{pmatrix} 14 & -10 \\ 2 & -8 \end{pmatrix}$$
$$= \begin{pmatrix} 2-14, & 0-(-10) \\ 4-2, & -5-(-8) \end{pmatrix} = \begin{pmatrix} -12, & 10 \\ 2, & 3 \end{pmatrix}$$

Matrix Multiplication:

Recall: When a matrix B multiplies a vector $\mathbf{x}$, it transforms $\mathbf{x}$ into the vector $B\mathbf{x}$.

If this vector is then multiplied in turn by a matrix A, the resulting vector is $A(B\mathbf{x})$. Thus $A(B\mathbf{x})$ is produced from $\mathbf{x}$ by a composition of linear transformations.

I. The dot product of $\vec{v}_1 = (a_1\ a_2 \cdots a_p) \in \mathbb{R}^p$

& $\vec{v}_2 = (b_1, b_2, \cdots b_p) \in \mathbb{R}^p$

$$\vec{v}_1 \cdot \vec{v}_2 = a_1 b_1 + a_2 b_2 + \cdots a_p b_p.$$

II. $A \in \mathbb{R}^{m \cdot n}$ (# col of A), $B \in \mathbb{R}^{p \cdot q}$ (# rows of B.)

If $A \cdot B$ (A multiplied by B) is defined,

only if $n = p$

Only if the # cols of the first matrix

= # rows of the second matrix

$A = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $A \cdot B$ ✓

$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}_{2 \cdot 3}$ $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}_{2 \cdot 2}$ $A \cdot B$ (X)

col of A $\neq$ # rows of B.

Review.

I. Dot product between $\vec{a} = (a_1\ a_2\ \cdots\ a_n)$

& $\vec{b} = (b_1\ b_2\ \cdots\ b_n)$: $a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$

II. $A \in \mathbb{R}^{m \cdot n}$, $B \in \mathbb{R}^{p \cdot q}$

$A \cdot B$ is defined only if $n = p$

(# of cols of the first matrix A

= # of rows of the second matrix B)

III. $(A \cdot B)$ is still a matrix, dimension is : $m \cdot q$

# of rows of $A \cdot B$ = # of rows of A = m

# of cols of $A \cdot B$ = # of cols of B = q.

IV. $(AB)_{ij}$ = dot product of ith row of A

& the jth col of B.

We would like to represent this composite mapping as multiplication by a single matrix, denoted by AB, so that

$$A(B\mathbf{x}) = (AB)\mathbf{x}.$$

Definition: If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $\mathbf{b}_1, \cdots, \mathbf{b}_p$, then the product AB is an $m \times p$ matrix whose columns are $A\mathbf{b}_1, \cdots, A\mathbf{b}_p$, i.e.

Remark:

- Each column of AB is a linear combination of the columns of A using weights from the corresponding column of B.

- AB has the same number of rows as A and the same number of columns as B.

Example 2: If a matrix A is 4×5 and the product AB is 4×6, what is the size of B?

Row-column rule for computing AB:
If the product AB is defined, then the entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B.

If $(AB)_{ij}$ denotes the (i, j)-entry in AB, and if A is an $m \times n$ matrix, then

Example 3: Let $A = \begin{bmatrix} 2 & 0 \\ 4 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$, compute the product AB by the row–column rule for computing AB.

$A \cdot B_{2\cdot 2} =$

the 1st row, the 1st col

$[2, 0]$ dot products $[1, -2]$

1st row of A   1st col of B

$2 \cdot 1 + 0 \cdot (-2) = 2$,

the 1st row, the 2nd col

$[2, 0] \cdot [2, 1]$

1st row of A   2nd col of B.

$2 \cdot 2 + 0 \cdot 1 = 4$

$[4, -5] \cdot [1, -2]$

the 2nd row of A   the 1st col of B

$4 \cdot 1 + (-5)(-2) = 14$

$[4, -5] \cdot [2, 1]$

$4 \cdot 2 - 5 \cdot 1$

$= 3$.

$2 \cdot 2$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\cdot 2} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\cdot 3}$$

1st row of A   1st col of B   2nd col of B

$$A \cdot B = \begin{bmatrix} [1, 2] \text{ dot } [1, 4] & [1, 2] \text{ dot } [2, 5] & [1, 2] \text{ dot } [3, 6] \\ 1\cdot 1 + 2\cdot 4, & 1\cdot 2 + 2\cdot 5, & 1\cdot 3 + 2\cdot 6 \\ [3, 4] \cdot \text{dot } [1, 4] & [3, 4] \text{ dot } [2, 5] & [3, 4] \text{ dot } [3, 6] \\ 3\cdot 1 + 4\cdot 4, & 3\cdot 2 + 4\cdot 5, & 3\cdot 3 + 4\cdot 6. \end{bmatrix}_{2\cdot 3}$$

2nd row of A   1st col of B

\# rows of A

\# cols of B.

Theorem: Properties of Matrix Multiplication:

Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

a. $A(BC) = (AB)C$ (associative law of multiplication)

b. $A(B + C) = AB + AC$ (left distributive law)

c. $(B + C)A = BA + CA$ (right distributive law)

d. $r(AB) = (rA)B = A(rB)$ for any scalar r

e. $I_m A = A = AI_n$ (identity for matrix multiplication)

$$I_m = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & & 0 \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}_{m \cdot m}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3 \cdot 3}$$

WARNINGS:

1. In general, $AB \neq BA$.
2. The cancellation laws do *not* hold for matrix multiplication. That is, if $AB = AC$, then it is *not* true in general that $B = C$.
3. If a product AB is the zero matrix, you *cannot* conclude in general that either $A = 0$ or $B = 0$.

Power of a Matrix:

$$A^k = \underbrace{A \cdot A \cdot A \cdots A}_{k \text{ copies of } A}.$$

If A is an $n \times n$ matrix and if k is a positive integer, then A^k denotes the product of k copies of A.

Example 4: Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$. Compute AD and DA. Find a 2×2 matrix $B \neq I_2$, such that $AB = BA$.

The transpose of a Matrix:

A^t

Given an $m \times n$ matrix A, the transpose of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

Let A and B denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T = A$

b. $(A+B)^T = A^T + B^T$

c. For any scalar r, $(rA)^T = rA^T$

d. $(AB)^T = B^T A^T$

Remark: (d)The transpose of a product of matrices equals the product of their transposes in the reverse order.

Example 6: Let $\mathbf{u} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$, compute $\mathbf{u}^T\mathbf{v}$, $\mathbf{v}^T\mathbf{u}$, $\mathbf{u}\mathbf{v}^T$, $\mathbf{v}\mathbf{u}^T$.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad A^t = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$