Section 2.2 The Inverse of a Matrix

Definitions:

1. An $n \times n$ matrix A is said to be invertible if there is an $n \times n$ matrix C such that identity matrix

$$CA = I \quad \text{and} \quad AC = I \qquad CA = A \cdot C = I$$

   where $I = I_n$ is the $n \times n$ identity matrix. C is an inverse of A, and C is uniquely determined by A. The unique inverse is denoted by A^{-1}.

2. A matrix that is not invertible is sometimes called a singular matrix, and an invertible matrix is called a nonsingular matrix.

Theorem: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and
det(A) ≠ 0

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible.

The quantity $ad - bc$ is called the ___determinant___ of A,

$$\det(A) = ad - bc$$

Example 1: Find the inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$.

$\det(A) = 1 \cdot 5 - 2 \cdot 3 = -1$

$A^{-1} = \frac{1}{-1} \begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix} = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$

Theorem: If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Example 2: Use the inverse of the matrix A to solve the system

$$\begin{aligned} x_1 + 2x_2 &= 1 \\ 3x_1 + 5x_2 &= 3 \end{aligned}$$

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}_{2\cdot 2}$$

$2 \to$ # eqns

$2 \to$ # of unknowns

$$x = A^{-1} \cdot b = \underbrace{\begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}}_{2\cdot 2} \cdot \underbrace{\begin{pmatrix} 1 \\ 3 \end{pmatrix}}_{2\cdot 1}$$

$$= \begin{pmatrix} (-5, 2) \text{ dot } (1, 3) \\ -5\cdot 1 + 2\cdot 3 = 1 \\ (3, -1) \text{ dot } (1, 3) \\ 3\cdot 1 - 1\cdot 3 = 0 \end{pmatrix}_{2\cdot 1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Theorem:

1. If A is an invertible matrix, then A^{-1} is invertible and
$$(A^{-1})^{-1} = A$$

2. If A and B are $n \times n$ matrices, then so is AB, and the inverse of AB is the product of the inverses of A and B in the reverse order. That is,
$$(AB)^{-1} = B^{-1}A^{-1}$$

3. If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. That is,
$$(A^T)^{-1} = (A^{-1})^T$$

Elementaty Matrices: An elementary matrix is one that is obtained by performing a single elementary row operation on an identity matrix.

Example 3: Let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}, A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.
Compute E_1A, E_2A and E_3A, and describe how these products can be obtained by elementary row operations on A.

E_1: go from the $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3\cdot 3} = I_3$

$R_3 = R_3 - 4R_1$ (replacement).

E_2. interchange R_1 & R_2 of I_3

E_3: $R_3 = 5 \cdot R_3 \rightarrow$ (comes from I_3)

Remark: If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA, where the $m \times m$ matrix E is created by performing the same row operation on I_m.

$$E_1 A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$= \begin{pmatrix} a & b & c \\ d & e & f \\ -4a+g, & -4b+h, & -4c+i \end{pmatrix}$$

Theorem: An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

Algorithm for finding A^{-1}:

Row reduce the augmented matrix $[A\ I]$. If A is row equivalent to I, then $[A\ I]$ is row equivalent to $[I\ A^{-1}]$. Otherwise, A does not have an inverse.

$E_1 = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$, $E_1 \cdot I$

Example 4: Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ if it exists.

$$\left[\underbrace{\begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix}}_{A} \middle| \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I_2}\right] \xrightarrow[3 \to 0]{R_2 = R_2 - 3R_1} \left[\begin{matrix} 1 & 2 \\ 0 & \underbrace{4-2\cdot 3}_{-2} \end{matrix} \middle| \begin{matrix} 1 & 0 \\ 0-3 & 1-3\cdot 0 \end{matrix}\right]$$

$E_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$$\xrightarrow[-2 \to 0]{R_1 = R_1 + R_2} \left[\begin{matrix} 1 & 0 \\ 0 & -2 \end{matrix} \middle| \begin{matrix} 1-3 & 0+1 \\ -3 & 1 \end{matrix}\right]$$

$E_3 = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

$$\xrightarrow[-2 \to 1]{R_2 = (-\frac{1}{2}) \cdot R_2} \left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \middle| \underbrace{\begin{matrix} -2 & 1 \\ 3/2 & -1/2 \end{matrix}}_{A^{-1}}\right]$$

Section 2.3 Characterizations of Invertible Matrices

Theorem:

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.
b. A is row equivalent to the $n \times n$ identity matrix.
c. A has n pivot positions.
d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e. The columns of A form a linearly independent set.
f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
h. The columns of A span $\mathbb{R}^n$.
i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
j. There is an $n \times n$ matrix C such that $CA = I$.
k. There is an $n \times n$ matrix D such that $AD = I$.
l. A^T is an invertible matrix.