Review

$A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$, Find the inverse of A.

$$\left[\underbrace{\begin{matrix} 2 & 1 \\ 5 & 3 \end{matrix}}_{A} \;\middle|\; \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I_2}\right]$$

change A to the I_2

(if A is invertible, this is similar to find $\text{rref}(A)$.

$$\xrightarrow[R_1 \,\&\, R_2]{\text{interchange}} \left[\begin{matrix} 5 & 3 \\ 2 & 1 \end{matrix} \;\middle|\; \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}\right] \xrightarrow[5 \to 1]{R_1 = R_1 - 2R_2} \left[\begin{matrix} 1 & 1 \\ 2 & 1 \end{matrix} \;\middle|\; \begin{matrix} -2 & 1 \\ 1 & 0 \end{matrix}\right]$$

$$\xrightarrow[2 \to 0]{R_2 = R_2 - 2R_1} \left[\begin{matrix} 1 & 1 \\ 0 & -1 \end{matrix} \;\middle|\; \begin{matrix} -2 & 1 \\ 5 & -2 \end{matrix}\right] \xrightarrow[1 \to 0]{R_1 = R_1 + R_2} \left[\begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} \;\middle|\; \begin{matrix} 3 & -1 \\ 5 & -2 \end{matrix}\right]$$

$R_2 = R_2 \cdot (-1)$

$-1 \to 1$

$$\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I_2} \left|\; \underbrace{\begin{matrix} 3 & -1 \\ -5 & 2 \end{matrix}}_{A^{-1}} \right.$$

Section 2.8 Subspaces of $\mathbb{R}^n$

$H \subseteq \mathbb{R}^n$

Definition: A subspace of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has three properties:

1. The zero vector is in H. $\vec{0} \in H$, $\mathbb{R}^3$, $H \subseteq \mathbb{R}^3$, $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
2. For each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u}+\mathbf{v}$ is in H.
3. For each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

Remark: A subspace is closed under addition and scalar multiplication.

Example 1: If $\mathbf{v}_1$ and $\mathbf{v}_2$ are in $\mathbb{R}^n$ and $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, then H is a subspace of $\mathbb{R}^n$. Verify this.

all possible linear combination of $\vec{v}_1$ & $\vec{v}_2$

$\{v = c_1\vec{v}_1 + c_2\vec{v}_2$, c_1 & c_2 are arbitrary scalars$\}$.

Remark: The set of all linear combinations of $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.

Definitions: The column space of a matrix A is the set Col (A) of all linear combinations of the columns of A.

$A = \{\vec{v}_1, \vec{v}_2\}$, $\vec{v}_1$ & $\vec{v}_2$ are columns of A.

$\text{Col}(A) = \text{span}\{\vec{v}_1, \vec{v}_2\}$.

Definition: The null space of a matrix A is the set $\underbrace{\text{Null}(A)}_{\text{notation}}$ of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

$A = [\vec{v}_1, \dots, \vec{v}_n]_{m \cdot n}$, $\vec{v}_1, \dots \vec{v}_n \in \mathbb{R}^m$

Theorem:

- The column space of an $m \times n$ matrix is a subspace of $\mathbb{R}^m$.
- The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.

$[A]_{m \cdot n}\, x = \vec{0} \in \mathbb{R}^n$, $x \in \mathbb{R}^n$

null $(A) = \{$ all the $\vec{x} \in \mathbb{R}^n$ s.t. $A \cdot x = 0 \}$

Example 3: For the matrix A below, find a nonzero vector in Null A and a nonzero vector in Col A.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \end{bmatrix}_{2 \cdot 3}$$

$B = \{ \vec{b}_1, \vec{b}_2 \dots \vec{b}_p \}$

$\vec{b}_1 \dots \vec{b}_p$ are linearly indep.

B (1)

Definition: A basis for $\boxed{\text{a subspace } H \text{ of } \mathbb{R}^n}$ is a linearly independent set in H that spans H.

(2)

The columns of $n \times n$ identity matrix $\mathbf{e}_1, \cdots, \mathbf{e}_n$ is called the standard basis for $\mathbb{R}^n$.

cols of I_n.

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots, \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$\mathbb{R}^3$, $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\{ \vec{e}_1\ \vec{e}_2\ \vec{e}_3 \}$ is the standard basis of $\mathbb{R}^3$

Example 4: Find a basis for (1) the null spcae and (2) the column space of the matrix

$$A = \begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{bmatrix}$$

Theorem: The pivot columns of a matrix A form a basis for the column space of A.

eg 4

$$\begin{pmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{pmatrix} \xrightarrow[4 \to 1]{R_1 = R_1 - R_3} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{pmatrix}$$

$$\xrightarrow[6 \to 0]{R_2 = R_2 - 6R_1} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -5 & 6 \\ 3 & 4 & 8 & -3 \end{pmatrix}$$

$$\xrightarrow[3 \to 0]{R_3 = R_3 - 3R_1} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -5 & 6 \\ 0 & 1 & 5 & -6 \end{pmatrix} \xrightarrow[1 \to 0]{R_3 = R_2 + R_3}$$

$$\begin{pmatrix} \boxed{1} & 1 & 1 & 1 \\ 0 & \boxed{-1} & -5 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\underbrace{x_1 \quad x_2}_{\text{basis}} \quad \underbrace{x_3 \quad x_4}_{\text{free}}$$

$$\begin{cases} x_3 = s \\ x_4 = t \end{cases}$$

row 2, $\quad -1x_2 - 5x_3 + 6x_4 = 0$

$$x_2 = -5s + 6t$$

row 1, $\quad x_1 + x_2 + x_3 + x_4 = 0$

$$x_1 = -x_2 - s - t$$

$$= 5s - 6t - s - t$$

$$= 4s - 7t$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4s - 7t \\ -5s + 6t \\ s + 0t \\ 0s + t \end{pmatrix} = \begin{pmatrix} 4s \\ -5s \\ s \\ 0s \end{pmatrix} + \begin{pmatrix} -7t \\ 6t \\ 0t \\ t \end{pmatrix}$$

$$= s \begin{pmatrix} -4 \\ 5 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -7 \\ 6 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{null}(A) = \Big\{ s \underbrace{\begin{pmatrix} -4 \\ 5 \\ 1 \\ 0 \end{pmatrix}}_{\text{basis}} + t \underbrace{\begin{pmatrix} -7 \\ 6 \\ 0 \\ 1 \end{pmatrix}}_{\text{basis}}, \; s \,\&\, t \text{ are arbitrary in } \mathbb{R} \Big\}$$

basis of null(A)

$$\left\{ \begin{pmatrix} -4 \\ 5 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -7 \\ 6 \\ 0 \\ 1 \end{pmatrix} \right\}$$

basis of null(A) : parametric solution of the $Ax = 0$

① linear combination (span null (A))

② linearly indep.

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by the theorem,

basis of col (A) is $\left\{ \begin{pmatrix} 4 \\ 6 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ 5 \\ 4 \end{pmatrix} \right\}$

$\uparrow$ 1st col of A

$\uparrow$ 2nd col of A

eg 3. $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \end{pmatrix}$

For null (A). $Ax = 0$,

x_3 is free.

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \end{pmatrix} \xrightarrow{R_2 = R_2 - 2R_1} \begin{pmatrix} \boxed{1} & 2 & 0 \\ 0 & \boxed{-3} & 3 \end{pmatrix}$$

$2 \to 0$

x_1 x_2 $\to$ basic

$x_3 = s$, $s \in \mathbb{R}$.

we are solving a homg system.

From Row 2, $-3x_2 + 3x_3 = 0$

$$x_2 = x_3 = s$$

From Row 1, $x_1 + 2x_2 = 0$

$$x_1 = -2x_2 = -2s$$

solutions to the homog. system

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2s \\ s \\ s \end{pmatrix} = s\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$$\text{null}(A) = \left\{ s\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix},\ s \in \mathbb{R} \right\}$$

pick up any $s \neq 0$, eg: let $s = 1$.

$\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ is in null (A)

basis of null (A).

$\vec{u} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$,

1. $\vec{u}$ is linearly indep.
2. any vector in null (A) ($\vec{v}$)

$\vec{v} = \alpha \cdot \vec{u}$, for α is a scalar.