Section 2.9 Dimension and Rank

Definition: Suppose the set $\mathscr{B} = \{\mathbf{b}_1, \cdots, \mathbf{b}_p\}$ is a basis for a subspace H. For each $\mathbf{x}$ in H, the coordinates of $\mathbf{x}$ relative to the basis $\mathscr{B}$ are the weights $c_1, \cdots, c_p$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_p\mathbf{b}_p$, and the vector in $\mathbb{R}^p$,

$$[\mathbf{x}]_{\mathscr{B}} = \begin{bmatrix} c_1 \\ \cdots \\ c_p \end{bmatrix}$$

is called the coordinate vector of x (relative to B) or the B-coordinate vector of x

Remark: The main reason for selecting a basis for a subspace H, instead of merely a spanning set, is that each vector in H can be written in only one way as a linear combination of the basis vectors.

Example 1: The vector $\mathbf{x}$ is in a subspcae H with a basis $\mathscr{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, and

$$\mathbf{b}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$$

Find the $\mathscr{B}$-coordinate vector of $\mathbf{x}$.

Find $x_1, x_2 \in \mathbb{R}$ such that

$$x_1 \vec{b_1} + x_2 \vec{b_2} = \vec{x}$$

$\Leftrightarrow$ $A\vec{y} = \vec{x}$, $A = \{\vec{b_1} \ \vec{b_2}\}$, $\vec{y} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\Leftrightarrow$ augmented matrix format

$[\vec{b_1} \ \vec{b_2} \ \vec{x}] \longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$, from the 2^{nd} row $x_2 = 3$, from the 1^{st} row $x_1 = 2$ $\Rightarrow \vec{y} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Definition: The dimension of a nonzero subspace H, denoted by $\dim(H)$, is the number of vectors in any basis for H. The dimension of the zero subspace $\{\mathbf{0}\}$ is defined to be zero.

Remark: The space $\mathbb{R}^n$ has dimension n.

Definition: The rank of a matrix A, denoted by $\text{rank}(A)$, is the dimension of the column space of A.

Example 2: The echelon form of A is given, find basis for Col$A$ and Nul$A$, and then state the dimensions of these subspaces.

$$A = \begin{bmatrix} 1 & 3 & 2 & -6 \\ 3 & 9 & 1 & 5 \\ 2 & 6 & -1 & 9 \\ 5 & 15 & 0 & 14 \end{bmatrix} \sim \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} \boxed{1} & 3 & 3 & 2 \\ 0 & 0 & \boxed{5} & -7 \\ 0 & 0 & 0 & \boxed{5} \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

① def: col space of A is the set of all linear combinations of cols of A. (span{all cols of A})

② thm: the pivot cols of A form a basis for col(A)

col(A).

from the echelon form of A, the 1st, 3rd, 4th cols are pivot cols

$\overset{\text{thm}}{\Longrightarrow}$ basis $\left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -6 \\ 5 \\ 9 \\ 14 \end{pmatrix} \right\}$

$\dim(\text{col}(A)) = 3.$

The Rank Theorem: If a matrix A has n columns, then $\text{rank}A + \dim\text{Nul}(A) = n$.

$\text{rank}A$: $\dim(\text{col}(A))$

The Basis Theorem: Let H be a $p-$dimensional subspace of $\mathbb{R}^n$. Any linearly independent set of exactly p elements in H is automatically a basis for H. Moreover, any set of p elements of H that spans H is automatically a basis for H.

eg 2. null (A).

$$\text{null}(A) = \{ x, \ Ax = 0 \}$$

Since x_2 is free, $x_2 = s$, $s \in \mathbb{R}$

From the 4th row, ---

--- 3rd row, $5x_4 = 0 \Rightarrow x_4 = 0$

----- 2nd ---, $5x_3 - 7x_4 = 0$, $x_3 = 0$

----- 1st ---, $x_1 + 3x_2 + 3x_3 + 2x_4 = 0$

$$\Rightarrow x_1 = -3s$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3s \\ s \\ 0 \\ 0 \end{pmatrix} = s \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad s \in \mathbb{R}$$

$$\text{null}(A) = \left\{ s \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \ s \in \mathbb{R} \right\}$$

basis of null (A), $\left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$

$$\dim(\text{null}(A)) = 1$$

The Invertible Matrix Theorem: Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

1. The columns of A form a basis of $\mathbb{R}^n$.
2. Col$A = \mathbb{R}^n$.
3. dimCol$A = n$.
4. rank$A = n$.
5. Nul$A = \{\mathbf{0}\}$.
6. dimNul$A = 0$.

Example 3: True or False:

True a. If $\mathscr{B} = \{\mathbf{b}_1, \cdots, \mathbf{b}_p\}$ is a basis for a subspace H and if $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_p\mathbf{b}_p$, then $c_1, \cdots, c_p$ are the coordinates of $\mathbf{x}$ relative to the basis $\mathscr{B}$.

False b. Each line in $\mathbb{R}^n$ is a one-dimensional subspace of $\mathbb{R}^n$.

If a line is subspace of $\mathbb{R}^n$, this line must pass through the origin.

True c. The dimension of Col$A$ is the number of pivot columns in A.

True d. The dimensions of Col$A$ and Nul$A$ add up to the number of columns in A.

True e. If a set of p vectors spans a p-dimensional subspace H of $\mathbb{R}^n$, then these vectors form a basis for H.

Example 4: If the rank of a 9×8 matrix A is 7, what is the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$.

$\Updownarrow$

null(A) by the Rank thm,

$$\dim(\text{null}(A)) + \underbrace{\text{rank}(A)}_{7} = \#\text{ of col of } A = 8$$

$$\Rightarrow \dim(\text{null}(A)) = 1.$$