Section 3.1 Introduction to Determinants

Recall: The determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\det(A) = ad - bc$

Generalize to 3×3 matrix:

$$M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \xrightarrow[a \cdot R_3]{a \cdot R_2} \begin{bmatrix} a & b & c \\ ad & ae & af \\ ag & ah & ai \end{bmatrix} \xrightarrow[-g \cdot R_1 + R_3 \to R_3]{-d \cdot R_1 + R_2 \to R_2} \begin{bmatrix} a & b & c \\ 0 & ae - bd & af - cd \\ 0 & ah - bg & ai - cg \end{bmatrix}$$

$$\to \begin{bmatrix} a & b & c \\ 0 & A & B \\ 0 & C & D \end{bmatrix} \xrightarrow{A \cdot R_3} \begin{bmatrix} a & b & c \\ 0 & A & B \\ 0 & AC & AD \end{bmatrix} \xrightarrow{-C \cdot R_2 + R_3 \to R_3} \begin{bmatrix} a & b & c \\ 0 & A & B \\ 0 & 0 & AD - BC \end{bmatrix}$$

where $AD - BC = (ae - bd) - (ai - cg)(af - cd)(ah - bg)$

① determinant is only for the square matrix.

② det (A) is a scalar

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}_{n \cdot n} \qquad A_{12} = \begin{pmatrix} a_{21} & a_{23} & \cdots & a_{2n} \\ & & & \\ a_{n1} & a_{n3} & \cdots & a_{nn} \end{pmatrix}_{(n-1)(n-1)}$$

Definition:

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$

$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

$a_{13} \det(A_{13})$

A_{ij} : cross the ith row & jth col to get an $(n-1) \times (n-1)$ sub-matrix.

Example 1: Compute the determinant of

$$A = \begin{bmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{bmatrix}$$

$$\det(A) = \underset{a_{11}}{3} \cdot \underbrace{\det\begin{pmatrix} 3 & 2 \\ 5 & -1 \end{pmatrix}}_{A_{11}} - \underset{a_{12}}{0} \cdot \underbrace{\det\begin{pmatrix} 2 & 2 \\ 0 & -1 \end{pmatrix}}_{A_{12}} + \underset{a_{13}}{4} \cdot \underbrace{\det\begin{pmatrix} 2 & 3 \\ 0 & 5 \end{pmatrix}}_{A_{13}}$$

$$= 3 \cdot (3 \cdot (-1) - 2 \cdot 5) - 0 + 4 \cdot (2 \cdot 5 - 0 \cdot 3) = 1$$

The expansion of a 3×3 determinant can be remembered by the following device. Write a second copy of the first two columns to the right of the matrix, and compute the determinant by multiplying entries on six diagonals:

$$\begin{array}{c} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{array} \begin{array}{cc} - & - \quad - \\ a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ + & + \quad + \end{array}$$

Add the downward diagonal products and subtract the upward products. Use this method to compute the determinants in Exercises 15–18. ***Warning:*** *This trick does not generalize in any reasonable way to 4×4 or larger matrices.*

index of one entry

Remark: Given $A = [a_{ij}]$, the (i, j)-cofactor of A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

cross the ith row & jth col.

Then

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

Theorem:

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the ith row using the cofactors in (4) is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

The cofactor expansion down the jth column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Example 2: Use a cofactor expansion across the third row to compute det$A$, where

$$A = \begin{bmatrix} 6 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 2 & 0 & 0 & 0 \\ 8 & 3 & 1 & 8 \end{bmatrix}$$

row index col index

$$\det(A) = \underbrace{2}_{a_{31}} \cdot (-1)^{3+1} \cdot \det\begin{pmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 8 \end{pmatrix} + \underbrace{0}_{a_{32}} \cdot (-1)^{3+2} \cdot \underbrace{\det\begin{pmatrix} 6 & 0 & 5 \\ 1 & 2 & -5 \\ 8 & 1 & 8 \end{pmatrix}}_{A_{32}} = 0$$

3rd row 3rd col

$$+ \underbrace{0}_{a_{33}} \cdot (-1)^{3+3} \det\begin{pmatrix} 6 & 0 & 5 \\ 1 & 7 & -5 \\ 8 & 3 & 8 \end{pmatrix} = 0 + \underbrace{0}_{a_{34}} \cdot (-1)^{3+4} \cdot \det\begin{pmatrix} 6 & 0 & 0 \\ 1 & 7 & 2 \\ 8 & 3 & 1 \end{pmatrix} = 0$$

Theorem: If A is a triangular matrix, then det$A$ is the product of the entries on the main diagonal of A.

1st row of the sub matrix

1st col

$$\det(A) = 2 \cdot 1 \cdot \det\underbrace{\begin{pmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 8 \end{pmatrix}}_{B} = 2 \cdot \left(\underbrace{0}_{b_{11}} \cdot (-1)^{1+1} \cdot \det\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} = 0 \right.$$

$$\left. + \underbrace{0}_{b_{12}} \cdot (-1)^{1+2} \cdot \det\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} = 0 + 5 \cdot (-1)^{1+3} \cdot \det\begin{pmatrix} 7 & 2 \\ 3 & 1 \end{pmatrix} \right)$$

Theorem: $\det(A) = 2\cdot 1\cdot 5\cdot 1\cdot(7-2\cdot 3) = 2\cdot 5\cdot 1 = 10$

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the ith row using the cofactors in (4) is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

The cofactor expansion down the jth column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Example 2: Use a cofactor expansion across the third row to compute det$A$, where

$$A = \begin{bmatrix} 6 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 2 & 0 & 0 & 0 \\ 8 & 3 & 1 & 8 \end{bmatrix}$$

Theorem: If A is a triangular matrix, then det$A$ is the product of the entries on the main diagonal of A.