Section 3.2 Properties of Determinants

Theorem: (Row operations) Let A be a square matrix.

1. If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.
2. If two rows of A are interchanged to produce B, then $\det B = -\det A$.
3. If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.

Example 1: State the property of determinants for the following equations.

(a) $\begin{vmatrix} 2 & -6 & 4 \\ 3 & 5 & -2 \\ 1 & 6 & 3 \end{vmatrix} = 2 \begin{vmatrix} 1 & -3 & 2 \\ 3 & 5 & -2 \\ 1 & 6 & 3 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 3 & -4 \\ 2 & 0 & -3 \\ 5 & -4 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -4 \\ 0 & -6 & 5 \\ 5 & -4 & 7 \end{vmatrix}$

$A_0 \xrightarrow{R_1 = \frac{1}{2}R_1} A_1$, $\det(A_1) = \frac{1}{2}\det(A_0)$

$A_0 \xrightarrow{R_2 = R_2 - 2R_1} A_1$, $\det(A_0) = \det(A_1)$

Example 2: Find the determinants given $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5$.

(a) $\begin{vmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{vmatrix}$ $= 3 \cdot 5$

$A_0 \xrightarrow{R_2 = 3R_2} A_1$, $\det(A_1) = 3\det(A_0) = 3 \cdot 5 = 15$

(b) $\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix}$.

$A_0 \xrightarrow{R_2 = 2R_2} A_1 = \begin{pmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{pmatrix} \xrightarrow{R_2 = R_2 + R_1} B$

$\det(A_0) = \frac{1}{2}\det(A_1)$, $\det(A_1) = \det(B)$ $\Rightarrow \det(B) = 10.$

Remark: Suppose a square matrix A has been reduced to an echelon form U by row replacements and row interchanges. If there are r interchanges, then

(no scaling)

$$\det A = (-1)^r \det U$$

Notice that $\det U = u_{11} \cdot u_{22} \cdots u_{nn}$, which is the product of the diagonal entries of U. If A is invertible, the entries u_{ii} are all pivots. Otherwise, at least u_{nn} is zero. Thus

$$\det A = \begin{cases} (-1)^r \cdot \begin{pmatrix} \text{product of} \\ \text{pivots in } U \end{pmatrix} & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}_{2 \cdot 2}$$

$$\det(A) = 3$$

$$\det(\underbrace{2A}) = \underline{2 \cdot 2 \cdot 3}$$

$$\begin{pmatrix} 2a & 2b \\ 2c & 2d \end{pmatrix}$$

$$A \xrightarrow{R_1 = 2R_1} A_1 \xrightarrow{R_2 = 2R_2} 2A$$

$$A_1 = \begin{pmatrix} 2a & 2b \\ c & d \end{pmatrix}$$

$$2\det(A) = \det(A_1)$$

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}_{3 \cdot 3}, \quad \det(A) = 5$$

$$\det(2A) = 2 \cdot 2 \cdot 2 \cdot 5$$

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Example 3: Find the determinants by row reduction to echelon form.

$$\begin{vmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{vmatrix}$$

A_0

$A_0 \xrightarrow{R_2 = R_2 - 3R_1} \begin{pmatrix} 1 & 5 & -3 \\ 0 & -18 & 12 \\ 2 & 13 & -7 \end{pmatrix} = A_1 \xrightarrow{R_3 = R_3 - 2R_1} \begin{pmatrix} 1 & 5 & -3 \\ 0 & -18 & 12 \\ 0 & 3 & -1 \end{pmatrix} = A_2$

$\det(A_0) = \det(A_1)$, $\det(A_1) = \det(A_2)$

$\xrightarrow{R_2 \leftrightarrows R_3} \begin{pmatrix} 1 & 5 & -3 \\ 0 & 3 & -1 \\ 0 & -18 & 12 \end{pmatrix} = A_3 \xrightarrow{R_3 = R_3 + 6R_2} \begin{pmatrix} 1 & 5 & -3 \\ 0 & 3 & -1 \\ 0 & 0 & 6 \end{pmatrix} = A_4$

$\det(A_2) = -\det(A_3)$, $\det(A_3) = \det(A_4)$

$\Rightarrow \det(A_0) = -\det(A_4) = -1 \cdot 3 \cdot 6 = -18.$

Example 4: Use determinants to decide if the set of vectors is linearly independent.

$$\begin{bmatrix} 7 \\ -4 \\ -6 \end{bmatrix}, \begin{bmatrix} -8 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ -5 \end{bmatrix}$$

2.9 (thm) 2.3 (thm)

Theorem: A square matrix A is invertible if and only if $\det A \neq 0$.

Theorem: If A is an $n \times n$ matrix, then $\det A^T = \det A$.

Theorem: If A and B are $n \times n$ matrices, then $\det AB = (\det A)(\det B)$.