Section 3.3 Cramer's Rule, Volume, And Linear Transforamtions

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For any $n \times n$ matrix A and any $\mathbf{b}$ in $\mathbb{R}^n$, let $A_i(\mathbf{b})$ be the matrix obtained from A by replacing column i by the vector $\mathbf{b}$

$$A_i(\mathbf{b}) = [\mathbf{a}_1 \ \cdots \ \mathbf{b} \ \cdots \ \mathbf{a}_n]$$

replace the ith col of A by $\vec{b}$

Theorem: (Cramer's Rule): Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \quad i = 1, 2, \cdots, n \tag{1}$$

ith entry of the unknown vector $\vec{x}$

Proof:

used to solve

$$A\vec{x} = \vec{b}, \quad A \in \mathbb{R}^{n \cdot n}, \quad \vec{b} \in \mathbb{R}^n, \quad \vec{x} \in \mathbb{R}^n$$

(unknown)

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eg1. $A = \begin{pmatrix} 4 & 1 \\ 5 & 2 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 6 \\ 7 \end{pmatrix}$

$$\det(A) = 4 \cdot 2 - 1 \cdot 5 = 3$$

$$x_1 = \frac{\det(A_1(b))}{\det(A)} = \frac{\det\begin{pmatrix} 6 & 1 \\ 7 & 2 \end{pmatrix}}{3} = \frac{6 \cdot 2 - 7}{3} = \frac{5}{3}$$

Example 1: Use Cramer's rule to solve the system

$$4x_1 + x_2 = 6$$
$$5x_1 + 2x_2 = 7$$

$$x_2 = \frac{\det(A_2(b))}{\det(A)} = \frac{\det\begin{pmatrix} 4 & 6 \\ 5 & 7 \end{pmatrix}}{3} = \frac{4 \cdot 7 - 6 \cdot 5}{3} = \frac{-2}{3}$$

A formula for A^{-1}: For an invertible $n \times n$ matrix A, the j-th column of A^{-1} is a vector $\mathbf{x}$ that satisfies

$$A\mathbf{x} = \mathbf{e}_j$$

the i-th entry of $\mathbf{x}$ is the (i, j)-entry of A^{-1}. By Cramer's rule,

$$\{(i, j)\text{-entry of } A^{-1}\} = x_i = \frac{\det A_i(\mathbf{e}_j)}{\det A} \tag{2}$$

Recall: A_{ji} denotes the submatrix of A formed by deleting row j and column i, thus

$$\cancel{\det A_i(\mathbf{e}_j) =} (-1)^{i+j} \det A_{ji} \overset{\text{def}}{=} C_{ji}$$

where C_{ji} is a cofactor of A.

Thus

$A^{-1} \cdot \det(A) = \mathrm{adj}(A)$

$\overset{I}{\cancel{A^{-1} A}} \cdot \det(A) = \mathrm{adj}(A) \cdot A$

$I \cdot \det(A) = \mathrm{adj}(A) \cdot A$

$$A^{-1} = \frac{1}{\det A} \underbrace{\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}}_{\text{adjugate } (A)} \tag{3}$$

C_{ij} here are the cofactors of ij the entry of matrix A.

The matrix of cofactors on the right side of (3) is called the adjugate (or classical adjoint) of A, denoted by adj$A$.

Theorem (An inverse formula): Let A be an invertible $n \times n$ matrix, then

$$A^{-1} = \frac{1}{\det A} \mathrm{adj} A$$

$I \cdot \det(A) = \mathrm{adj}(A) \cdot A$

Example 2: Find the inverse of $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$

$C_{11} = (-1)^{1+1} \cdot 2 \cdot \det\begin{pmatrix} -1 & 1 \\ 4 & -2 \end{pmatrix}$

$C_{12} = (-1)^{1+2} \cdot 1 \cdot \det\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$

↳ 1st row 2nd col entry cofactor of matrix A.

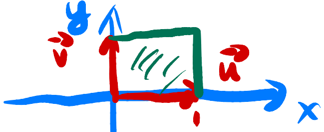
$\mathrm{adj}(A) = \begin{pmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{pmatrix}$

$\begin{pmatrix} \det(A) & 0 & 0 \\ 0 & \det(A) & 0 \\ 0 & 0 & \det(A) \end{pmatrix} = \begin{pmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{pmatrix}$

$\Rightarrow \det(A) = 14$

$A^{-1} = \frac{1}{14} \cdot \mathrm{adj}(A)$

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\cdot 2}$ $\quad \vec{u} \ \ \vec{v}$



Theorem: If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$. If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.

Remark: Let $\mathbf{a}_1$ and $\mathbf{a}_2$ be nonzero vectors. Then for any scalar c, the area of the parallelogram determined by $\mathbf{a}_1$ and $\mathbf{a}_2$ equals the area of the parallelogram determined by $\mathbf{a}_1$ and $\mathbf{a}_2 + c\mathbf{a}_1$.

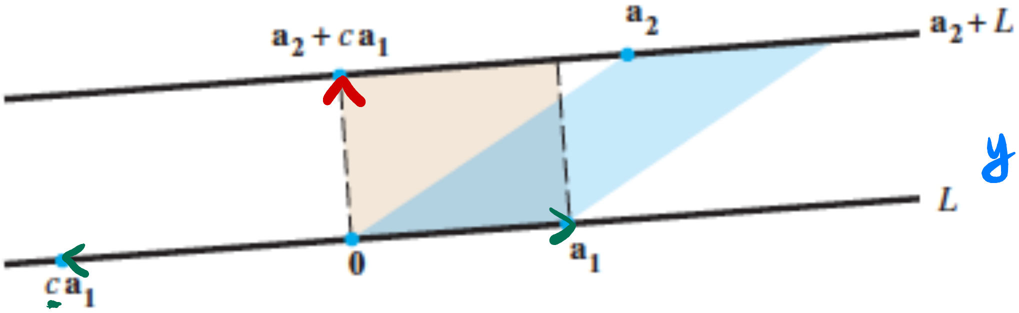
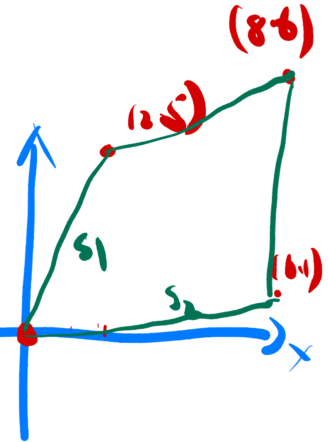


FIGURE 2 Two parallelograms of equal area.



Example 3: Find the area of the parallelogram whose vertices are (~~−1,0~~ −2,−2), (0,~~5~~ 3), (~~1,−4~~ 4,−1), (~~2,1~~ 6,4).

subtract (−2,2) from all 4 vertices

⟺ shift the parallelogram to the origin

⟺ the area does not change.

(0,0), (2,5) (6,1) (8,6)

$$\text{area} = |\det(A)|$$
$$= \left|\det\begin{pmatrix} 2 & 6 \\ 5 & 1 \end{pmatrix}\right|$$
$$= |2\cdot 1 - 5\cdot 6| = 28$$

Example 4: Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(1,4,0)$, $(-2,-5,2)$ and $(-1,2,-1)$

$T(x) = A \overset{\downarrow}{\cdot} x$ standard matrix multiplication

Theorem: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then

$$\{\text{area of } T(S)\} = |\det A| \cdot \text{area of } S\}$$

If T is determined by a 3×3 matrix A, and if S is a parallelepiped in $\mathbb{R}^3$R3, then

$$\{\text{volume of } T(S)\} = |\det A| \cdot \text{volume of } S\}$$

suppose we have a parallelepiped S.

& a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ (A)

Q: Is that possible $T(S) = AS$ is a parallelogram ?

A: Yes. $$AS = \underbrace{\begin{pmatrix} x & x & x \\ x & x & x \\ 0 & 0 & 0 \end{pmatrix}}_{B} \leftarrow$$ entire one row $= 0$.

Q: what properties should A satisfy ?

$$\det(B) = 0$$

$$B = AS$$

$$0 = \det(B) = \det(AS) = \det(A) \cdot \det(S)$$

$\Rightarrow \det(A) = 0 \iff A$ is singular.