Section 4.1 Vector Spaces and Subspaces

Definition (Vector space):

A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms (or rules) listed below.[1] The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d.

addition

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} + \mathbf{v}$, is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
4. There is a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$. → $\vec{0}$ in V is unique
5. For each $\mathbf{u}$ in V, there is a vector $\underset{w}{-\mathbf{u}}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$. ($u + w = 0$) → $-u$ (w) is also unique.

scalar multiplication

6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$.
10. $1\mathbf{u} = \mathbf{u}$. ($1 \in \mathbb{R}$)

Examples:

$\mathbb{P}_n$: set of all polynomials of degree less than or equal to n.

$p(t) = a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n$, $a_0, a_1 \dots a_n$ are coeffs, they are $\mathbb{R}$.
t is dependent variable.

① degree of $p(t)$ is the highest power of t.

② If $a_0 = a_1 = \dots = a_n = 0$, $p(t) = 0$, zero polynomial, (zero in the $\mathbb{P}_n$)

addition: $q(t) = b_0 + b_1 t + \dots b_n t^n$, $p + q = (a_0 + b_0) + (a_1 + b_1)t + \dots + (a_n + b_n)t^n$

scalar multiplication, $c \in \mathbb{R}$

$cp = a_0 c + a_1 c t + a_2 c t^2 + \dots + a_n c t^n$.

Facts: For each $\mathbf{u}$ in V and scalar c,

$$0\mathbf{u} = \mathbf{0}$$

($0 \in \mathbb{R}$; $\mathbf{0}$: 0 vector in the v.s.)

$$c\mathbf{0} = \mathbf{0}$$

$$-\mathbf{u} = (-1)\mathbf{u}$$

Example 1: Let V be the first quadrant in the xy-plane; that is, tlet $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0 \right\}$ (1) If $\mathbf{u}$ and $\mathbf{v}$ are in V, is $\mathbf{u}+\mathbf{v}$ in V?

$\vec{u} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $x_1 \geq 0$, $y_1 \geq 0$ $\quad \vec{u} + \vec{v} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$

$\vec{v} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, $x_2 \geq 0$, $y_2 \geq 0$ $\quad$ b/c $x_1 + x_2 \geq 0$, $y_1 + y_2 \geq 0$

$\Rightarrow \vec{u} + \vec{v} \in V$, V is closed under addition.

(2) Find a specific vector $\mathbf{u}$ in V and a specific scalar c such that $c\mathbf{u}$ is not in V.

$u = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in V$

$c = -1 \quad cU = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \notin V$, not closed under scalar multiplication.

Definition (Subspace):

$H \subseteq V$

A **subspace** of a vector space V is a subset H of V that has three properties:

a. The zero vector of V is in H.

b. H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u}+\mathbf{v}$ is in H.

c. H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

thm: a subspace is also a vector space.

Examples:

1. The set consisting of only the zero vector in a vector space V is a subspace of V, called the zero subspace and written as $\{\mathbf{0}\}$.

2. Let $\mathbb{P}$ be the set of all polynomials with real coefficients, with operations in $\mathbb{P}$ defined as for functions. Then $\mathbb{P}$ is a subspace of the space of all real-valued functions defined on $\mathbb{R}$.

$\mathbb{R}^3 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix}, x, y, z \in \mathbb{R} \right\}$

3. The vector space $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$ because $\mathbb{R}^2$ is not even a subset of $\mathbb{R}^3$.

$\hookrightarrow \mathbb{R}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix}, x \in \mathbb{R}, y \in \mathbb{R} \right\}$

4. A plane in $\mathbb{R}^3$ not through the origin is not a subspace of $\mathbb{R}^3$.

$(n \geq 2)$

Example 2: Determine if the given set is a subspace of $\mathbb{P}_n$ for an appropriate value of n.

$V = \{$ (1) All polynomials of the form $\mathbf{p}(t) = at^2$, where a is in $\mathbb{R}$ $\}$

① If we set $a = 0$, $p(t) = 0 \cdot t^2 = \vec{0} \Rightarrow \vec{0}$ is in this set.

② $p = a_1 t^2$, $a_1 \in \mathbb{R}$, $q = a_2 t^2$, $a_2 \in \mathbb{R}$, $p, q \in V$, $p + q = (a_1 + a_2) t^2 \in V$ ✓

③ $p = a_1 t^2$, $c \in \mathbb{R}$, $cp = \underbrace{a_1 c}_{\in \mathbb{R}} t^2 \Rightarrow cp \in V \Rightarrow$ closed.

$V: \{$ (2) All polynomials of the form $\mathbf{p}(t) = a + t^2$, where a is in $\mathbb{R}$ $\}$

No, zero vector of $\mathbb{P}_n$ is not in $V \Rightarrow V$ is not a subspace.

Theorem: If $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$ are in a vector space V, then Span$\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ is a subspace of V. We call Span$\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ the subspace spanned (or generated) by $\mathbf{v}_1, \cdots, \mathbf{v}_p$.

Example 3: Let W be a set of all vectors of the form $\begin{bmatrix} 2s+4t \\ 2s \\ 2s-3t \\ 5t \end{bmatrix}$. Show that W is a subspace of $\mathbb{R}^4$.

$$\begin{pmatrix} 2s+4t \\ 2s+0 \\ 2s-3t \\ 0 \cdot s+5t \end{pmatrix} = \begin{pmatrix} 2s \\ 2s \\ 2s \\ 0 \end{pmatrix} + \begin{pmatrix} 4t \\ 0 \\ -3t \\ 5t \end{pmatrix}$$

$$= s \underbrace{\begin{pmatrix} 2 \\ 2 \\ 2 \\ 0 \end{pmatrix}}_{u \in \mathbb{R}^4} + t \underbrace{\begin{pmatrix} 4 \\ 0 \\ -3 \\ 5 \end{pmatrix}}_{v \in \mathbb{R}^4}$$

$$= \text{span}\{\vec{u}, \vec{v}\} \overset{\text{thm}}{\Rightarrow} \text{this is a subspace of } \mathbb{R}^4.$$

Example 4: Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.

(1) Is $\mathbf{w}$ in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? How many vectors are in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

(2) How many vectors are in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

(3) Is $\mathbf{w}$ in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?