Section 4.2 Null Spaces, Column Spaces and Linear Transformations

The null space of a matrix: The null space of an $m \times n$ matrix A, written as Nul A, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation,

$$\text{null}(A) = \text{Nul}A = \{\mathbf{x} : \ \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$$

Theorem: The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

Example 1 (An explicit description of Nul A): Find the spanning set for the null space of the matrix $\begin{bmatrix} 1 & -3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$.

Example 2: Either use an appropriate theorem to show that the given set, W , is a vector space, or find a specific example to the contrary.

(1) $V = \left\{ \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\in \mathbb{R}^3} : \ a+b+c=2 \right\}$ $\quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad 0+0+0 \neq 2, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \notin V$

$V =$ (2) $\left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : 3a + b = c,\ a + b + 2c = 2d \right\}$

find $a, b, c, d \in \mathbb{R}$ s.t.

$\Leftrightarrow \begin{cases} 3a + b - c + 0 \cdot d = 0 \\ a + b + 2c - 2d = 0 \end{cases}$

coeff matrix

$\Leftrightarrow \underbrace{\begin{pmatrix} 3 & 1 & -1 & 0 \\ 1 & 1 & 2 & -2 \end{pmatrix}_{2 \cdot 4}}_{A} \cdot \underbrace{\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}}_{x} = 0$

$V = \{x,\ Ax = 0\}$

$V = \text{null}(A)$

$\Rightarrow V$ is a subspace of $\mathbb{R}^4$

by the thm (4.1)

$\Rightarrow V$ is also a vector space.

The column space of a matrix: The column space of an $m \times n$ matrix A, written as Col A, is the set of all linear combinations of the columns of A. If $A = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_n]$, then

$$\text{Col}A = \text{Span}\{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$$

$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $x_1 \cdots x_n \in \mathbb{R}$

Theorem: The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

$$\text{Col}A = \{\mathbf{b} : \ \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbb{R}^n\}$$

$\underbrace{Ax}_{\text{matrix form}} \Leftrightarrow \underbrace{\vec{u}_1 \cdot x_1 + \vec{u}_2 \cdot x_2 + \cdots + \vec{u}_n \cdot x_n}_{\text{vector form}}$

Remark: The column space of an $m \times n$ matrix A is all of $\mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^m$.

$\Updownarrow$

$\text{col}(A) = \mathbb{R}^m$

Example 3: Find A such that the given set is Col A

$V = \left\{ \begin{bmatrix} b - c \\ 2b + 3d \\ b + 3c - 3d \\ c + d \end{bmatrix} : \ b, c, d \text{ real} \right\}$

$\begin{pmatrix} b - c + 0 \cdot d \\ 2b + 0 \cdot c + 3d \\ b + 3c - 3d \\ 0 \cdot b + c + d \end{pmatrix} \otimes \left\{ \begin{pmatrix} b \\ 2b \\ b \\ 0 \end{pmatrix} + \begin{pmatrix} -c \\ 0 \\ 3c \\ c \end{pmatrix} + \begin{pmatrix} 0 \cdot d \\ 3d \\ -3d \\ d \end{pmatrix} \right\}$

$b \cdot \underbrace{\begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}}_{u_1} \quad c \underbrace{\begin{pmatrix} -1 \\ 0 \\ 3 \\ 1 \end{pmatrix}}_{u_2} \quad d \underbrace{\begin{pmatrix} 0 \\ 3 \\ -3 \\ 1 \end{pmatrix}}_{u_3}$

V is the linear combination of $\vec{u}_1\ \vec{u}_2\ \vec{u}_3$

$V = \text{span}\{\vec{u}_1\ \vec{u}_2\ \vec{u}_3\}$

$A = [\vec{u}_1\ \vec{u}_2\ \vec{u}_3]$

Contrast Between Nul A and Col A for an $m \times n$ Matrix A

Nul A	Col A
1. Nul A is a subspace of $\mathbb{R}^n$.	**1.** Col A is a subspace of $\mathbb{R}^m$.
2. Nul A is implicitly defined; that is, you are given only a condition ($A\mathbf{x} = \mathbf{0}$) that vectors in Nul A must satisfy.	**2.** Col A is explicitly defined; that is, you are told how to build vectors in Col A.
3. It takes time to find vectors in Nul A. Row operations on $[\,A \quad \mathbf{0}\,]$ are required.	**3.** It is easy to find vectors in Col A. The columns of A are displayed; others are formed from them.
4. There is no obvious relation between Nul A and the entries in A.	**4.** There is an obvious relation between Col A and the entries in A, since each column of A is in Col A.
5. A typical vector $\mathbf{v}$ in Nul A has the property that $A\mathbf{v} = \mathbf{0}$.	**5.** A typical vector $\mathbf{v}$ in Col A has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in Nul A. Just compute $A\mathbf{v}$.	**6.** Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in Col A. Row operations on $[\,A \quad \mathbf{v}\,]$ are required.
7. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	**7.** Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
8. Nul $A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	**8.** Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.

linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$, $T(x) = Ax$

Definition: A linear transformation T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W, such that

1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$, for all $\mathbf{u}, \mathbf{v}$ in V
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in V and all scalars c

$\Rightarrow T(c\vec{u} + d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$, $c, d \in \mathbb{R}$, $\vec{u}, \vec{v} \in V$

The kernel or null space of T is the set of all $\mathbf{u}$ in V such that $T(\mathbf{u}) = \mathbf{0}$. $\ker(T) \subseteq V$ (domain of T) (null(A))
The range of T is the set of all vectors in W of the form $T(\mathbf{x})$ for some $\mathbf{x}$ in V.

range$(T) = \{\vec{w},\ \vec{w} = T(\vec{x}),$ for $\vec{x} \in V\} \subseteq W$

Example 5: True or false.

a. ✓ The null space of A is the solution set of the equation $A\mathbf{x} = \mathbf{0}$. (col(A)

b. ✗ The null space of an $m \times n$ matrix is in $\mathbb{R}^m$. $\mathbb{R}^{(n)} \to$ # cols of A

c. ✓ The column space of A $\in \mathbb{R}^{m\times n}$ is the range of the mapping $\mathbf{x} \mapsto A\mathbf{x}$. thm, for some $x \in \mathbb{R}^n$

d. ✓ If the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$, then Col A is $\mathbb{R}^m$.

$\Downarrow$ for $\forall b \in \mathbb{R}^m$, you can find $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ s.t. $\vec{b} = \vec{a_1} x_1 + \dots \vec{a_n} x_n$

$\Leftrightarrow \mathbb{R}^m = \text{span}\{\vec{a_1} \dots \vec{a_n}\} = \text{col}(A)$

thm: $\boxed{\text{e. ✓ The kernel of a linear transformation is a vector space.}}$