Section 4.3 Linear Independent Sets, Bases

An indexed set of vectors $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ in V is said to be linearly independent if the vector equation

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \quad (1)$$

has only the trivial solution, $c_1 = 0, \cdots, c_p = 0$.

① a set of 2 vectors is linear dep. iff one of the vector is the multiple of the other.

The set $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ in V is said to be linearly dependent if (1) has nontrivial solution.

That is, if there are some weights, $c_1, \cdots, c_p$, not all zero, such that (1) holds.

thm

An indexed set $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ of two or more vectors, with $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if some $\mathbf{v}_j$ (with $j > 1$) is
a linear combination of the preceding vectors $\mathbf{v}_1, \cdots, \mathbf{v}_{j-1}$.

Example 1: Let $p_1(t) = 1$, $p_2(t) = t$, $p_3(t) = t^2$, $p_4(t) = 4t - 1$, $p_5(t) = 2t^2 - 5t + 3$,

(1) is the set $\{p_1(t), p_2(t)\}$ linearly dependent?

NO. $p_2(t) = t = t \cdot 1 = \boxed{t}\, p_1(t)$

↳ is not a const.

(2) is the set $\{p_1(t), p_2(t), p_4(t)\}$ linearly dependent?

$p_4(t) = 4t - 1 = 4p_2(t) - p_1(t)$

by the thm ⇒ linear dep.

(3) is the set $\{p_1(t), p_2(t), p_3(t)\}$ linearly dependent?

No

(4) is the set $\{p_1(t), p_2(t), p_3(t), p_5(t)\}$ linearly dependent?

$p_5(t) = 2t^2 - 5t + 3 = 2p_3(t) - 5p_2(t) + 3p_1(t)$

by thm ⇒ linear dep.

Definition: Let H be a subspace of a vector space V. An indexed set of vectors $\mathscr{B} = \{\mathbf{b}_1, \cdots, \mathbf{b}_p\}$ in V is a basis for H if

1. $\mathscr{B}$ is a linearly independent set, and
2. the subspace spanned by $\mathscr{B}$ coincides with H; that is

$$H = \text{Span}\{\mathbf{b}_1, \cdots, \mathbf{b}_p\}$$

Remark:
(1) Let A be an invertible $n \times n$ matrix, then the columns of A form a basis for $\mathbb{R}^n$.
(2) Let $\mathbf{e}_1, \cdots, \mathbf{e}_n$ be the columns of the $n \times n$ identity matrix I_n. The set $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ is called the standard basis for $\mathbb{R}^n$.
(3) $S = \{1, t, t^2, \cdots, t^n\}$ is a standard basis for $\mathbb{P}^n$. $\to$ polynomial of degree at most n.

Example 2: Find a basis for the set of vectors in $\mathbb{R}^2$ on the line $y = -3x$.

method 1.

set $= V = \left\{ \begin{pmatrix} x \\ -3x \end{pmatrix}, x \in \mathbb{R} \right\}$ $\quad$ x, $y = -3x$

on the line $y = -3x$

$\begin{pmatrix} x \\ -3x \end{pmatrix} = x \begin{pmatrix} 1 \\ -3 \end{pmatrix}, x \in \mathbb{R}$

$\Rightarrow \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ is the basis

method 2.

$3x + y = 0$, x & y are the entries of the vector. $\Leftrightarrow$ null(A) $\quad A = [3, \; 1]_{1 \times 2} \to$ 2 unknowns

$\Leftrightarrow$ Solve the eqn $3x + y = 0$, x & y are unknowns

are equat!!

Example 3: Find a basis for the set of functions defined on $\mathbb{R}$, $\{t, \sin t, \sin 2t, \sin t \cos t\}$

Ref $(A) = [\boxed{3} \; \; 1]$ $\quad 1 \to s$

$x_1 = s$

$3x + y = 0 \Rightarrow x = -\frac{s}{3}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{s}{3} \\ s \end{pmatrix} = s \begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix}$

$\Rightarrow$ basis $\begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix}$

The Spanning Set Theorem: Let $S = \{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ be a set in V, and $H = \text{Span}\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$.

a. If one of the vectors in S, $\mathbf{v}_k$ is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.

b. If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.

Remark: :

Eg. $S = \{t, \sin(t), \sin(2t), \sin(t)\cos(t)\}$

step 1, $\sin(2t) \overset{\text{formula.}}{=} 2\underbrace{\sin(t)\cos(t)}_{\text{4th vector of the set}}$

step2 $S_1 = \{t \ \sin(t) \ \sin(2t)\}$ is the new set.

If S_1 is linear indep, $\overset{\text{def of}}{\underset{\text{basis}}{\Rightarrow}}$ S_1 is the basis of S.

(i) pick up 3 numbers (3 = # of vectors in the set)

$$\begin{array}{l} t = \frac{\pi}{2}, \\ t = \pi, \\ t = \frac{\pi}{4}, \end{array} \quad \begin{pmatrix} \frac{\pi}{2}, & \sin(\frac{\pi}{2}) = 1, & \sin(2 \cdot \frac{\pi}{2}) = 0 \\ \pi, & \sin(\pi) = 0 & \sin(2\pi) = 0 \\ \frac{\pi}{4} & \sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} & \sin(2 \cdot \frac{\pi}{4}) = 1 \end{pmatrix} = A$$

(ii) If $\det(A) \neq 0$, or, A is invertible

$\Rightarrow$ S_1 is linear indep.

* the method can be used only to show linear indep.

$\Rightarrow$ the basis $= \{t, \sin(t), \sin(2t)\}$ b/c $\begin{cases} \text{① it can span } S \\ \text{② linear indep.} \end{cases}$

Bases for Nul A and Col A:

Theorem: The pivot columns of a matrix A form a basis for Col A.

Example 3: Find a basis for the set of vectors in set of real numbers $\mathbb{R}^3$ in the plane $x - 2y + 3z = 0$.

$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\Leftrightarrow$ Find x, y, z s.t. $x - 2y + 3z = 0$ → 1 eqn

$\Leftrightarrow$ null (A), $A = [1, -2, 3]$ $1 \cdot 3$ → 3 unknown $x\ y\ z$.

coeff matrix

$\text{Ref}(A) = [\boxed{1}, \underbrace{-2, 3}_{\text{free}}]$

$y = s$ & $z = t$ are free.

$x - 2y + 3z = 0$

$\Leftrightarrow \quad x = 2s - 3t \quad \Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2s - 3t \\ s \\ t \end{pmatrix}$

$$= \begin{pmatrix} 2s \\ s \\ 0 \end{pmatrix} + \begin{pmatrix} -3t \\ 0 \\ t \end{pmatrix}$$

$$= s \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

$\Rightarrow$ basis $= \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$