Section 4.5 The Dimension of a Vector Space

Recall: Let H be a subspace of a vector space V. An indexed set of vectors $\mathscr{B} = \{\mathbf{b}_1, \cdots, \mathbf{b}_p\}$ in V is a basis for H if

1. $\mathscr{B}$ is a linearly independent set, and
2. the subspace spanned by $\mathscr{B}$ coincides with H; that is

$$H = \text{Span}\{\mathbf{b}_1, \cdots, \mathbf{b}_p\}$$

additional vector $\vec{u} \in V$
$S = \{\vec{b_1}, \vec{b_2} \cdots \vec{b_p}, \vec{u}\}$ is linear dep.
↳ $\vec{u}$ can be written as a linear combination of $\vec{b_1} \cdots \vec{b_p}$ (b/c they are basis)

Theorem: If a vector space V has a basis $\mathscr{B} = \{\mathbf{b}_1, \cdots, \mathbf{b}_p\}$, then any set in V containing more than n vectors must be linearly dependent.

Theorem: If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

Definition: If V is spanned by a finite set, then V is said to be finite-dimensional and the dimension of V, written as dim V, is the number of vectors in a basis for V.

The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero.

dim(col(A)) = rank(A)

If V is not spanned by a finite set, then V is said to be infinite-dimensional.

The Dimensions of Nul A and Col A: The dimension of Nul A is the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$, and the dimension of Col A is the number of pivot columns in A.

nullity = dim(null(A))

Example 1: Find a basis for the subspace and state the dimension. (1) $\left\{ \begin{bmatrix} s-2t \\ s+t \\ 3t \end{bmatrix} : s, t \text{ in } \mathbb{R} \right\}$

thm in 4.2

$$\underbrace{\begin{pmatrix} s-2t \\ s+t \\ 3t \end{pmatrix}}_{y} = \underbrace{\begin{pmatrix} 1 & -2 \\ 1 & 1 \\ 0 & 3 \end{pmatrix}}_{A}{}_{3\cdot 2} \cdot \underbrace{\begin{pmatrix} s \\ t \end{pmatrix}}_{x} \Leftrightarrow \left\{ y : y = Ax, x = \begin{pmatrix} s \\ t \end{pmatrix}, s, t \in \mathbb{R} \right\}$$

$\Downarrow$ col(A)

the basis of the original set = basis of col (A)

$$\text{Ref}(A) = \begin{pmatrix} \boxed{1} & -2 \\ 0 & \boxed{3} \\ 0 & 0 \end{pmatrix} \to \text{pivots cols are the basis}$$

$$\Rightarrow \text{basis} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \right\} \quad \dim = 2.$$

(2) $\{(a, b, c) : a + 2b - c = 0\}$

Example 2: Find the dimension of the subspace of all vectors in set of real numbers $\mathbb{R}^5$ whose first and fifth entries are equal.

$$\left\{ \begin{pmatrix} a \\ b \\ c \\ d \\ a \end{pmatrix}, \ a\, b\, c\, d \in \mathbb{R} \right\} \qquad \begin{pmatrix} a \\ b \\ c \\ d \\ a \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ 0 \\ 0 \\ a \end{pmatrix} + \begin{pmatrix} 0 \\ b \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ c \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ d \\ 0 \end{pmatrix}$$

$$= a \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}^{u_1} + b \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}^{u_2} + c \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}^{u_3} + d \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}^{u_4}$$

① any vector in the set is a linear combination of $\vec{u}_1 \dots \vec{u}_4$

② $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ are linearly indep. $\Rightarrow$ they are basis.

Theorem: Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and

$$\dim H \leq \dim V$$

The Basis Theorem: Let V be a p-dimensional vector space, $p \geq 1$.

- Any linearly independent set of exactly p elements in V is automatically a basis for V.

  $= \dim(V)$! $\Rightarrow$ set spans V.

- Any set of exactly p elements that spans V is automatically a basis for V.

  $= \dim(V)$ $\Rightarrow$ set is linear indep.

Example 3: The first four Hermite polynomials are 1, $2t$, $-2+4t^2$, and $-12t+8t^3$. Show that the first four Hermite polynomials form a basis of $\mathbb{P}_3$.