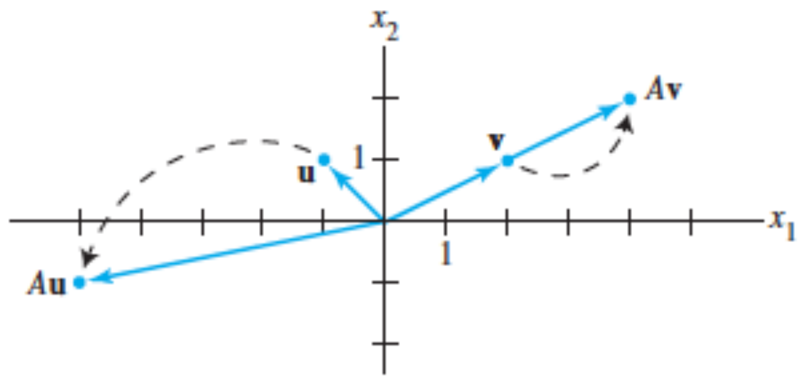
Section 5.1 Eigenvectors and Eigenvalues

Example 1: Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. The images of $\mathbf{u}$ and $\mathbf{v}$ under multiplication by A are shown in Figure .



Note that, $A\mathbf{v}$ is just $2\mathbf{v}$. So A only stretches $\mathbf{v}$.

eig-pair = (λ, x)

Definition: An eigenvector of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ.
A scalar λ is called an eigenvalue of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$, such an $\mathbf{x}$ is called an eigenvector corresponding to λ.

Example 2: Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, show that 7 is an eigenvalue of A and find the corresponding eigenvectors.

$Ax = \lambda x$

$Ax - \lambda I x = 0$, I is the identity matrix

$(A - \lambda I) \cdot x = 0$

the eigenvector exists, if the homog eqn has a non-trivial sol.
(eig vector $\neq 0$)

$\det(A - \lambda I) = 0$
↳ characteristic eqn of A.

the sols of $\det(A - \lambda I) = 0$ are just the eig-vals of A.

Step2.

The eigen vectors of λ are just non-trivial sol of $(A - \lambda I)x = 0$.

eg: $A = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}$

$\det(A - \lambda I) = 0 \quad (*)$

step 1, we need to find λ such that $(*)$ is true.

$$\det \underbrace{\begin{pmatrix} 1-\lambda, & 6 \\ 5, & 2-\lambda \end{pmatrix}}_{A - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} = 0$$

$(\lambda-1)\cdot(\lambda-2) - 30 = 0$

$\lambda^2 - 2\lambda - \lambda + 2 - 30 = 0$

$\lambda^2 - 3\lambda - 28 = 0$

$(\lambda-7)(\lambda+4) = 0$

eig-vals: $\lambda_1 = 7$ & $\lambda_2 = -4$.

Step 2.

$\lambda_1 = 7$

solve

$(A - 7I)\cdot x = 0 \quad (**)$

all non-zero solutions of $(**)$ are eigen-vectors of $\lambda = 7$

solve the following

$A - \lambda I = \begin{pmatrix} -6 & 6 \\ 5 & -5 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} -6 & 6 \\ 0 & 0 \end{pmatrix}$

set $x_2 = s$, $\Rightarrow x_1 = s$

$x = s \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $s \in \mathbb{R}$, $s \neq 0$

$\lambda_2 = -4$.

$A - \lambda_2 I = \begin{pmatrix} 5 & 6 \\ 5 & 6 \end{pmatrix}$

set $x_2 = s$, $x_1 = -\frac{6}{5}s$

$\Rightarrow x = s \begin{pmatrix} -\frac{6}{5} \\ 1 \end{pmatrix}$, $s \in \mathbb{R}$, $s \neq 0$

eg: $A = \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}$

step 1.

solve for λ

$\det(A - \lambda I) = 0$ $\quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\det \begin{pmatrix} 3-\lambda & 0 \\ 2 & 1-\lambda \end{pmatrix} = 0$$

$$(\lambda - 3)(\lambda - 1) = 0$$

$\Rightarrow \lambda_1 = 1, \lambda_2 = 3$

are eig-val of A.

step 2.

$\lambda_1 = 1$

Find $x (\neq 0)$ such that

$$(A - \lambda_1 I) x = 0$$

$$A - \lambda_1 I = \begin{pmatrix} 3-1, & 0 \\ 2, & 1-1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$$

set $x_2 = s$ to be free.

$\Rightarrow x_1 = 0$

sol $\left\{ x = s \begin{pmatrix} 0 \\ 1 \end{pmatrix}, s \in \mathbb{R} \right\}$

$\hookrightarrow$ eigen space of $\lambda_1 = 1$.

$\lambda_2 = 3$

Find x s.t.

$$(A - \lambda_2 I) x = 0$$

$$A - \lambda_2 I = \begin{pmatrix} 3-3 & 0 \\ 2 & 1-3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & -2 \end{pmatrix}$$

$x_2 = s, \Rightarrow x_1 = s$

$\left\{ x = s \begin{pmatrix} 1 \\ 1 \end{pmatrix}, s \in \mathbb{R} \right\}$

$\hookrightarrow$ eig-space of $\lambda_2 = 3$.

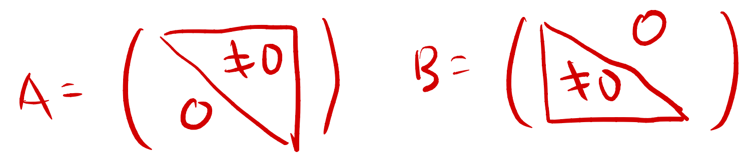
Remark: λ is an eigenvalue of an $n \times n$ matrix A if and only if the equation

$$(A - \lambda I)\mathbf{x} = \mathbf{0} \tag{1}$$

has a nontrivial solution.
The set of all solutions of (1) is just the null space of the matrix $A - \lambda I$, so this set is a subspace of $\mathbb{R}^n$.
It is called the eigenspace of A corresponding to λ. , which consists of the zero vector and all the eigenvectors corresponding to λ.

e'g space of λ = null space of $(A - \lambda I)$

Example 3: Let $A = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}$. The eigenvalues of A are 1 and 3. Find the eigenspace corresponding to each eigenvalue.



Theorem: The eigenvalues of a triangular matrix are the entries on its main diagonal.
Remark: The matrix A has an eigenvalue of 0 if and only if the equation

$$A\mathbf{x} = 0\mathbf{x} \tag{2}$$

has a nontrivial solution. But (2) is equivalent to $A\mathbf{x} = \mathbf{0}$, which has a nontrivial solution if and only if A is not invertible. Thus ________________________________
0 is an eigenvalue of A if and only if A is not invertible.

Example 4: Find the eigenvalues of the matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 2 \end{bmatrix}$ → upper triangular matrix

by the thm

Find λ s.t.

$\det(A - \lambda I) = 0$ $\quad (-\lambda)\cdot\det\begin{pmatrix} 3-\lambda & 4 \\ 0 & 2-\lambda \end{pmatrix} = 0$ $\quad \Rightarrow \lambda_1 = 0 \quad \lambda_2 = 3 \quad \lambda_3 = 2.$

$\det\begin{pmatrix} -\lambda & 0 & 0 \\ 0 & 3-\lambda & 4 \\ 0 & 0 & 2-\lambda \end{pmatrix} = 0$ $\quad (-\lambda)(\lambda-3)\cdot(\lambda-2) = 0$

diagonal entries

$\Rightarrow \lambda_1 = 0 \quad \lambda_2 = 3 \quad \lambda_3 = 2$

Theorem: If $\mathbf{v}_1, \cdots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \cdots, \lambda_r$

of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \cdots, \mathbf{v}_r\}$ is linearly independent.