Section 5.2 The Characteristic Equation

Recall: Suppose a square matrix A has been reduced to an echelon form U by row replacements and row interchanges. If there are r row interchanges, then

$$\det A = (-1)^r \det U$$

Notice that $\det U = u_{11} \cdot u_{22} \cdots u_{nn}$, which is the product of the diagonal entries of U. If A is invertible, the entries u_{ii} are all pivots. Otherwise, at least u_{nn} is zero. Thus

$$\det A = \begin{cases} (-1)^r \cdot \begin{pmatrix} \text{product of} \\ \text{pivots in } U \end{pmatrix} & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases}$$

The Invertible Matrix Theorem (continued): Let A be an $n \times n$ matrix. Then A is invertible if and only if

1. The number 0 is not an eigenvalue of A.
2. The determinant of A is not zero.

Theorem: Let A and B be $n \times n$ matrices.

a. A is invertible if and only if $\det A \neq 0$.

b. $\det AB = (\det A)(\det B)$.

c. $\det A^T = \det A$.

d. If A is triangular, then $\det A$ is the product of the entries on the main diagonal of A.

e. A row replacement operation on A does not change the determinant. A row interchange changes the sign of the determinant. A row scaling also scales the determinant by the same scalar factor.

Definition: The scalar equation

$$det(A - \lambda I) = \mathbf{0} \tag{1}$$

$Ax = \lambda x$

$Ax - \lambda I x = 0$

$(A - \lambda I)x = 0$

↳ has non-trivial sol

is called the characteristic equation of A.
A scaler λ is an eigenvalue of an $n \times n$ matrix A if and only if λ satisfies the characteristic equation (1).

$\Rightarrow \det(A - \lambda I) = 0$

$x^2(\lambda-1)=0$, $\lambda_1=\lambda_2=0$, mul $(\lambda=0)=2$

$\lambda_3=1$ mul$(\lambda_3=1)=1$

the degree of this polynomial in λ (characteristic) = the size of A.

Remark: $\det(A-\lambda I)$ is a polynomial in λ.
It can be shown that if A is an $n\times n$ matrix, then $\det(A-\lambda I)$ is a polynomial of degree n called the characteristic polynomial of A.
The (algebraic) multiplicity of an eigenvalue λ is its multiplicity as a root of the characteristic equation.

= # of times, λ appears as root in the characteristic eqn.

Example 2: Find the characteristic polynomial of the matrix $\begin{bmatrix}3&0&0\\2&1&4\\1&0&4\end{bmatrix}$, using either a cofactor expansion or the special formula for 3×3 determinants described in section 3.1. Find the eigenvalues and their multiplicities.

$$\det\begin{pmatrix}3-\lambda&0&0\\2&1-\lambda&4\\1&0&4-\lambda\end{pmatrix}=0$$

$\lambda_1=3$, $\lambda_2=1$, $\lambda_3=4$

multiplicity for all of them $=1$.

$$(3-\lambda)\det\begin{pmatrix}1-\lambda&4\\0&4-\lambda\end{pmatrix}=0$$

$$(3-\lambda)(1-\lambda)(4-\lambda)=0$$

$$\underbrace{(3-\lambda)(\lambda-1)(\lambda-4)=0}_{\text{characteristic eqn.}}$$

Example 3: List the real eigenvalues of $A=\begin{bmatrix}3&0&0&0\\6&2&0&0\\0&3&2&0\\2&3&3&-5\end{bmatrix}$, repeated according to their multiplicities.

A is a lower triangular matrix.

		multiplicity
(thm 5.1) $\Rightarrow$	$\lambda_1=3$	1
	$\lambda_2=\lambda_3=2$	2
	$\lambda_3=-5$	1

(diagonal entries are e.vals)

$P^{-1}AP=B$

$AP=PB$

$A=PBP^{-1}$

Definition: If A and B are $n\times n$ matrices, then A is similar to B if there is an invertible matrix P such that $P^{-1}AP=B$, or, equivalently, $A=PBP^{-1}$.
We say that A and B are similar. Changing A into $P^{-1}AP$ is called a similarity transformation.

Theorem: If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

Remark:

① If A & B have the same characteristic polynomials, A & B are not neccesarily similar to each other.

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

② similarity $\neq$ row equivalent.