Section 5.3 Diagonalization

Example 1: Let $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, find D^2, D^3, and D^k.

Review: $Ax = \lambda x$

① eig-vector: $x \neq 0$, $\lambda \in \mathbb{R}$

② $\det(A - \lambda I) = 0$ (characteristic equation)

↳ is a polynomial of degree n

③ Find $x \neq 0$ s.t. $(A - \lambda I)x = 0$.

④ eig space of λ $= \text{null}(A - \lambda I)$

⑤ the eigenvector corresponding to $\boxed{\text{distinct eigenvalues}}$ are linear independent.

⑥ $A \sim B$, there exists an invertible matrix P s.t. $A = PBP^{-1}$, $B = P^{-1}AP$

Remark: If $A = PDP^{-1}$ for some invertible P and diagonal D, then A^k is easy to compute.

Definition: A square matrix is diagonalizable if A is similar to a diagonal matrix.

The Diagonalization Theorem: An $n \times n$ matrix A is diagonalizable if only if A has n linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if

- the columns of P are n linearly independent eigenvectors of A.
- the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

Example 2: Use the Diagonalization Theorem to find the eigenvalues of A and a basis for each eigenspace. $A = \begin{bmatrix} 3 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 1 & -3 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -3 & 1 & 9 \\ -1 & 0 & 3 \end{bmatrix}$.

Section 5.3 Diagonalization

Example 1: Let $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, find D^2, D^3, and D^k.

then, suppose $D = \begin{pmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{pmatrix}_{n \times n}$

$$\underbrace{D^k}_{\underbrace{D \cdot D \cdots D}_{k \text{ copies of } D}} = \begin{pmatrix} a_1^k & & & 0 \\ & a_2^k & & \\ & & \ddots & \\ 0 & & & a_n^k \end{pmatrix}_{n \times n}$$

Remark: If $A = PDP^{-1}$ for some invertible P and diagonal D, then A^k is easy to compute.

$A^k = \underbrace{PD\cancel{P^{-1}} \cdot \cancel{P}D\cancel{P^{-1}}\cancel{P}DP^{-1} \cdots \cancel{P}DP^{-1}}_{k \text{ copies}} = P \cdot D^k \cdot P^{-1}$

there exists an invertible P & a diagonal matrix D s.t. $A = PDP^{-1}$

Definition: A square matrix is diagonalizable if A is similar to a diagonal matrix.

The Diagonalization Theorem: An $n \times n$ matrix A is diagonalizable if only if A has n linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if

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Example 2: Use the Diagonalization Theorem to find the eigenvalues of A and a basis for each eigenspace.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix} = \underbrace{\begin{array}{ccc} \vec{v_1} & \vec{v_2} & \vec{v_3} \\ \end{array} \begin{bmatrix} 3 & 0 & -1 \\ 0 & 1 & -3 \\ 1 & 0 & 0 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ -3 & 1 & 9 \\ -1 & 0 & 3 \end{bmatrix}}_{P^{-1}}.$$

$\vec{v_1}$ → 1st col of P

its corresponding e-value will be 1st diagonal entry of D.

$\vec{v_2}$ are linearly indep e-vectors of A

$\vec{v_3}$

$\lambda_1 = 3$

$\lambda_2 = 4$

$\lambda_3 = 3$

Example 3: Diagonalize the matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$.

Solution: Step 1. Find the eigenvalues of A.

Step 2. Find three linearly independent eigenvectors of A.

Step 3. Construct P from the vectors in step 2.

Step 4. Construct D from the corresponding eigenvalues.

eg3.

$$\det(A - \lambda I) = 0$$

$$\det \begin{pmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{pmatrix} = 0$$

$$-(\lambda-1)(\lambda+2)^2 = 0 \quad \Rightarrow \quad \lambda_1 = 1, \ \lambda_2 = \lambda_3 = -2.$$

$\lambda_1 = 1 \qquad (A - \lambda_1 I)x = 0, \quad x \neq 0$

$$A - \lambda I = \begin{pmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{pmatrix} \xrightarrow{\text{EROs}} \begin{pmatrix} \underline{3} & 3 & 0 \\ 0 & \underline{3} & 3 \\ 0 & 0 & \boxed{0} \end{pmatrix}$$

set $x_3 = s, \Rightarrow x_2 = -s, \quad x_1 = s$

$$x = \underbrace{s \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}_{\vec{u}_1,\ s=1}, \quad s \in \mathbb{R},\ s \neq 0$$

$\lambda_2 = -2$

$$A - \lambda_2 I = \begin{pmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{pmatrix} \xrightarrow{\text{EROs}} \begin{pmatrix} 3 & 3 & 3 \\ 0 & 0 & 0 \\ 0 & \underline{0} & \underline{0} \end{pmatrix}$$

$$\begin{array}{ccc} & s & t \end{array}$$

$x_2 = s \quad x_3 = t, \quad \Rightarrow \quad x_1 = -s - t$

$$x = \begin{pmatrix} -s-t \\ s \\ t \end{pmatrix} = s \underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}_{\vec{u}_2} + t \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{\vec{u}_3}, \quad s, t \in \mathbb{R},$$

$$P = \{\vec{u}_1\ \vec{u}_2\ \vec{u}_3\} = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$P = \begin{array}{c} \begin{pmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix} \\ \begin{matrix} \downarrow & \downarrow & \downarrow \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_2 \cdot 2 \end{matrix} \end{array}$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}_{3\cdot 3}$$

$$P = \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} -2 & & 0 \\ & 1 & \\ 0 & & -2 \end{pmatrix}$$

(⇒ linearly independent eigenvectors)

Theorem: An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Theorem:

$p \le n$

Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.

nullity of $\lambda_1 = 1$

a. For $1 \le k \le p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.

b. The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n, and this happens if and only if (*i*) the characteristic polynomial factors completely into linear factors and (*ii*) the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k. → $\dim(\text{null}(A-\lambda I))$ = multiplicity of λ

for egs: $1+2=3$ size

c. If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $\mathcal{B}_1, \ldots, \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.

nullity of $\lambda = -2$

{ linearly independent eigenvectors $n = n(\mathbb{R}^n)$ $\xRightarrow{\text{basis}}$ p is a basis (eigen-basis)

Example 4: A is a 5×5 matrix with two eigenvalues. One eigenspace is three-dimensional, and the other eigenspace is two dimensional. Is A diagonalizable? Why?

↳ Yes, thm (b)