# Section 5.4 Eigenvectors and Linear Transformations

**The Matrix of a Linear Transformation:** Let $V$ be an $n$-dimensional vector space, let $W$ be an $m$-dimensional vector space, and let $T$ be any linear transformation from $V$ to $W$. Let $\mathscr{B}$ and $\mathscr{C}$ be (ordered) bases for $V$ and $W$, respectively.

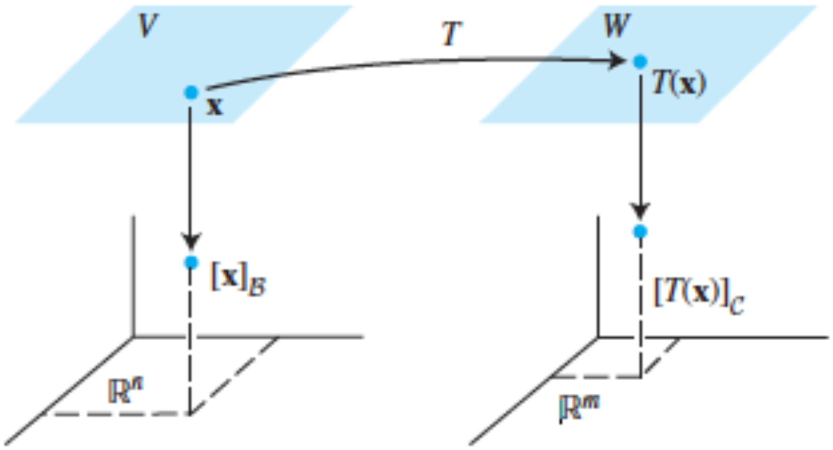


FIGURE 1 A linear transformation from $V$ to $W$.

Let $\{\mathbf{b}_1, \cdots, \mathbf{b}_n\}$ be the basis $\mathscr{B}$ for $V$, and $\{\mathbf{c}_1, \cdots, \mathbf{c}_m\}$ be the basis $\mathscr{C}$ for $W$.

suppose $T : V \longrightarrow V$ is a linear transformation.

$\mathcal{B} = \{\vec{b}_1, \vec{b}_2 \dots \vec{b}_n\}$ is basis.

Given $\vec{u} \in V$, $\vec{u} = r_1\vec{b}_1 + r_2\vec{b}_2 + \dots r_n\vec{b}_n$.

$[u]_{\mathcal{B}} \longrightarrow$ coord vectors relative to $\mathcal{B}$.

$$[u]_{\mathcal{B}} = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix}$$

Q: what is $[T(u)]_{\mathcal{B}}$

thm

$$[T(u)]_{\mathcal{B}} = \overbrace{\Big[ \underbrace{[T(\vec{b}_1)]_{\mathcal{B}},\ [T(\vec{b}_2)]_{\mathcal{B}}\ \dots\ [T(\vec{b}_n)]_{\mathcal{B}}}_{[T]_{\mathcal{B}}} \Big]_{n \times n}}^{\text{matrix representation of } T,\ \text{matrix for } T \text{ relative to basis } \mathcal{B}}$$

$$\cdot \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} \quad ([u]_{\mathcal{B}}$$

$$\Leftrightarrow [T(u)]_{\mathcal{B}} = [T]_{\mathcal{B}} \cdot [u]_{\mathcal{B}}$$

$$\begin{tikzcd} u \arrow[r, "T(u)"] \arrow[d] & T(u) \arrow[d] \\ {[u]_{\mathcal{B}}} \arrow[r, "\cdot [T]_{\mathcal{B}}"] & {[T(u)]_{\mathcal{B}}} \end{tikzcd}$$

eg 1. $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ is a basis for $V$ let: $T: V \to V$ be linear with property that $T(\vec{b}_1) = 3\vec{b}_1 - 2\vec{b}_2$ & $T(\vec{b}_2) = 4\vec{b}_1 + 7\vec{b}_2$

Find the matrix $M$ for $T$ relative to $\mathcal{B}$

$$[T]_{\mathcal{B}} = \Big[ [T(\vec{b}_1)]_{\mathcal{B}},\ [T(\vec{b}_2)]_{\mathcal{B}} \Big]$$

$$T(\vec{b}_1) = 3\vec{b}_1 - 2\vec{b}_2 \qquad = \left[ \begin{pmatrix} 3 \\ -2 \end{pmatrix} \begin{pmatrix} 4 \\ 7 \end{pmatrix} \right]$$

Suppose $T: V \to W$,

$\mathcal{B}_1 = \{\vec{b}_1, \vec{b}_2 \cdots \vec{b}_n\}$ $\quad \mathcal{B}_2 = \{\vec{p}_1\ \vec{p}_2 \cdots \vec{p}_m\}$

$\vec{u} \in V$, $\vec{u} = r_1 \vec{b}_1 + r_2 \vec{b}_2 + \cdots r_n \vec{b}_n$

$$[T(\vec{u})]_{\mathcal{B}_2} = \Big[ [T(\vec{b}_1)]_{\mathcal{B}_2} \cdots [T(\vec{b}_n)]_{\mathcal{B}_2} \Big]$$

$m \cdot n$

$\dim(W)$ $\quad \dim(V)$

**Example 1:** Let $\mathscr{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathscr{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases for vector spaces $V$ and $W$, respectively. Let $T : V \to W$ be a linear transformation with the property that

$$T(\mathbf{b}_1) = 3\mathbf{c}_1 - 3\mathbf{c}_2, \ \ , T(\mathbf{b}_2) = -2\mathbf{c}_1 + 5\mathbf{c}_2$$

Find the matrix for $T$ relative to $\mathscr{B}$ and $\mathscr{C}$.

$[Tx]_{\mathcal{B}}$

Review

$A = PDP^{-1}$

$P = [\ 1 \ 1 \ \cdots \ 1\ ]$

each col is a linear indep e. vectors

$D = \begin{pmatrix} \ddots \end{pmatrix}$

$T: V \longrightarrow V$

$\mathcal{B} = \{\vec{b}_1 \ldots \vec{b}_n\}$

$\vec{x} \in V \quad \vec{x} = c_1\vec{b}_1 + \ldots c_n\vec{b}_n$

$[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$

$Tx \in V$

thm

$[Tx]_{\mathcal{B}} = [T]_{\mathcal{B}} \cdot [x]_{\mathcal{B}}$

$[T]_{\mathcal{B}} = \Big[\ [T(\vec{b}_1)]_{\mathcal{B}}, \ [T(b_2)]_{\mathcal{B}} \ \ldots, \ [T(b_n)]_{\mathcal{B}} \Big]_{n \times n}$

**Example 2:** Let $\mathscr{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a basis for a vector space $V$ and let $T : V \to \mathbb{R}^2$ be a linear transformation with the property that

$$T(x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + x_3\mathbf{b}_3) = \begin{bmatrix} 2x_1 + x_2 - 5x_3 \\ 3x_1 - 2x_2 \end{bmatrix}$$

Find the matrix for $T$ relative to $\mathscr{B}$ and the standard basis for $\mathbb{R}^2$.

Eg3

(1) $\mathcal{B} = \{\underset{\vec{b}_1}{1}, \underset{\vec{b}_2}{t}, \underset{\vec{b}_3}{t^2}\}$

$[T]_{\mathcal{B}} = \Big[\ [T(\vec{b}_1)]_{\mathcal{B}} \ [T(\vec{b}_2)]_{\mathcal{B}} \ [T(\vec{b}_3)]_{\mathcal{B}} \Big]$

$T(b_1) = 0$

$b_1 = \overset{a_0}{1} + \overset{a_1}{0} \cdot t + \overset{a_2}{0} \cdot t^2$

$= 1 \cdot 1$
$+ 0 \cdot t$
$+ 0 \cdot t^2$

$T(b_1) = a_1 + 2a_2 t = 0 + 0 = 0 = 0 \cdot 1 + 0 \cdot t + 0 t^2$

$[T(b_1)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$T(b_2) = 1 \quad b_2 = t = \underset{a_0}{0} + \underset{a_1}{1} \cdot t + \underset{a_2}{0} \cdot t^2$

$[T(b_2)]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$T(b_3) = 2t \quad [T(b_3)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$

**Example 3:** Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be the transformation that maps a polynomial $\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2$ into the polynomial $a_1 + 2a_2 t$.

Now $[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

(1). Find the $\mathscr{B}$ matrix of T. where $\mathcal{B}$ is the std basis

(2). Verify: $[T(p)]_{\mathcal{B}} = [T]_{\mathcal{B}} [p]_{\mathcal{B}}$

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$p = a_0 \cdot 1 + a_1 t + a_2 t^2 \qquad [p]_{\mathcal{B}} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$

$T(p) = a_1 + 2a_2 t + 0 \cdot t^2 \qquad [T(p)]_{\mathcal{B}} = \begin{pmatrix} a_1 \\ 2a_2 \\ 0 \end{pmatrix}$

matrix multiplication $[T]_{\mathcal{B}} \cdot [p]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \overset{\text{check}}{=\!=} \begin{pmatrix} a_1 \\ 2a_2 \\ 0 \end{pmatrix}$ ✓

**Linear Transformation on $\mathbb{R}^n$:** In an applied problem involving $\mathbb{R}^n$, a linear transformation $T$ usually appears first as a matrix transformation, $\mathbf{x} \mapsto A\mathbf{x}$. If $A$ is diagonalizable, then there is a basis $\mathscr{B}$ for $\mathbb{R}^n$ consisting of eigenvectors of $A$.

assume $T: \mathbb{R}^n \to \mathbb{R}^n$, $T(x) = A \cdot x$

**Diagonal Matrix Representation**

Suppose $A = PDP^{-1}$, where $D$ is a diagonal $n \times n$ matrix. If $\mathcal{B}$ is the basis for $\mathbb{R}^n$ formed from the columns of $P$, then $D$ is the $\mathcal{B}$-matrix for the transformation $\mathbf{x} \mapsto A\mathbf{x}$.

the cols of $P$ form a basis of $\mathbb{R}^n$. $[T]_{\mathcal{B}}$

$$\begin{array}{ccc} x \in \mathbb{R}^n & \xrightarrow{\quad Ax \quad} & T(x) \in \mathbb{R}^n \\ {\scriptstyle P^{-1} \cdot x} \big\downarrow & & \big\downarrow \\ [x]_{\mathcal{B}} & \xrightarrow{[Tx]_{\mathcal{B}} = [T]_{\mathcal{B}} [x]_{\mathcal{B}}} & [Tx]_{\mathcal{B}} \end{array}$$

then

① $P$ is the basis for $\mathbb{R}^n$.

② & $A = PDP^{-1}$

$[T]_{\mathcal{B}} = D$.

**Example 4:** Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix}$. Find a basis $\mathscr{B}$

for $\mathbb{R}^2$ with the property that $[T]_{\mathscr{B}}$ is diagonal. $\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$

step 1. $\det(A - \lambda I) = 0$

$$\det\begin{pmatrix} -\lambda & 1 \\ -3 & 4-\lambda \end{pmatrix} = 0$$

$$(\lambda - 4)\cdot\lambda + 3 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0 \quad \Rightarrow \quad (\lambda-3)(\lambda-1) = 0, \Rightarrow \lambda_1 = 1 \quad \lambda_2 = 3.$$

step2.

$\lambda_1 = 1$, $\quad (A - \lambda_1 I)x = 0$

$$\begin{pmatrix} -1 & 1 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

set $x_2 = s \Rightarrow x_1 = s \qquad x = s\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad s \neq 0$

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\lambda_2 = 3.$ $\quad (A - \lambda_3 I)\cdot x = 0$

$$\begin{pmatrix} -3 & 1 \\ -3 & 1 \end{pmatrix}\cdot\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

set $x_2 = s \qquad -3x_1 + x_2 = 0$

$x_1 = \frac{1}{3}s \qquad \Rightarrow x = s\begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix}, \quad s \neq 0$

$$v_2 = \begin{pmatrix} 1/3 \\ 1 \end{pmatrix}$$

for $\mathbb{R}^2$ with the property that $[T]_{\mathscr{B}}$ is diagonal.

Now by thm 5.3,

$$\Rightarrow \quad P = \begin{pmatrix} 1 & 1/3 \\ 1 & 1 \end{pmatrix}$$

thm (5.1)

↳ eig vectors corresponding to distinct eig-values are linear indep.

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

Now from the thm, $\mathcal{B} = P$,

$$[T]_{\mathcal{B}} = D.$$

for $\mathbb{R}^2$ with the property that $[T]_{\mathscr{B}}$ is diagonal.

thm $T(x) = Ax$, $T : \mathbb{R}^n \to \mathbb{R}^n$.

(A & B) similar.

① If $A = PBP^{-1}$ & $\mathscr{B}$ (basis) of $\mathbb{R}^n = P$

$\Rightarrow [T]_{\mathscr{B}} = B$.

② If $P$ is the matrix whose cols form a basis of $\mathbb{R}^n$, $[T]_{\mathscr{B}} = P^{-1}AP$.

& $\mathscr{B} = P$, $P$ serves as the $\mathscr{B}$ of $\mathbb{R}^n$. $\overset{\shortparallel}{P}$